5.2.1 The two-level system

The most simple example for calculating probabilities and partition functions is the two level system with energy levels ϵ_1 and ϵ_2 and degeneracies $g_1 = g_2 = 1$ for classical particles, i.e. using the Maxwell-Boltzmann distribution function. A typical version for solving this problem is to introduce two new energy parameters $\bar{\epsilon} = (\epsilon_1 + \epsilon_2)/2$ and $\Delta\epsilon = (\epsilon_2 - \epsilon_1)/2$. According to Eq. (5.22) and Eq. (5.23) we find

$$n_1 = N \frac{e^{-\frac{\bar{\epsilon}-\Delta\epsilon}{kT}}}{e^{-\frac{\bar{\epsilon}-\Delta\epsilon}{kT}} + e^{-\frac{\bar{\epsilon}+\Delta\epsilon}{kT}}} = N \frac{e^{\frac{\Delta\epsilon}{kT}}}{e^{\frac{\Delta\epsilon}{kT}} + e^{-\frac{\Delta\epsilon}{kT}}} \tag{5.28}$$

and

$$n_2 = N \frac{e^{-\frac{\bar{\epsilon}+\Delta\epsilon}{kT}}}{e^{-\frac{\bar{\epsilon}-\Delta\epsilon}{kT}} + e^{-\frac{\bar{\epsilon}+\Delta\epsilon}{kT}}} = N \frac{e^{-\frac{\Delta\epsilon}{kT}}}{e^{\frac{\Delta\epsilon}{kT}} + e^{-\frac{\Delta\epsilon}{kT}}} \tag{5.29}$$

leading to

$$n_1 - n_2 = N \frac{e^{\frac{\Delta\epsilon}{kT}} - e^{-\frac{\Delta\epsilon}{kT}}}{e^{\frac{\Delta\epsilon}{kT}} + e^{-\frac{\Delta\epsilon}{kT}}} = N \tanh\left(\frac{\Delta\epsilon}{kT}\right) \quad . \tag{5.30}$$

Note: For the Maxwell-Boltzmann distribution any energy offset as well as any chemical potential contribution just cancel out, because they just show up as common factors in the nominator and denominator.
Eq. (5.30) can be used to model ferromagnetism. Let n_1 be the number of states with magnetic moment pointing "up" and n_2 the number of states with magnetic moment pointing "down". So $n_1 - n_2$ is proportional to the magnetization M of the ferromagnet. The energy of states with magnetic moments pointing "up" is proportional to $\Delta\epsilon \propto \mu_0(M+H)$ (H: external magnetic field) and the energy of states with magnetic moments pointing "down" is proportional to $-\Delta\epsilon \propto -\mu_0(M+H)$. Including some renormalization steps which will not be discussed here in detail Eq. (5.30) can thus be translated into

$$m = \tanh\left(\frac{m+h}{\tau}\right) \quad . \tag{5.31}$$

Easily one can see that $m = \pm 1$ is the saturation magnetization, and $\tau = 1$ is the Curie temperature, i.e. for $h = 0$ and $\tau < 1$ an intrinsic magnetization $m \neq 0$ is solution of Eq. (5.31).