3.12 The inverse temperature as an integrating factor

The phrase "integrating factor" originates from the theory for solving differential equations. The factor $1/T$ is called an integrating factor for δQ, since $dS = \delta Q/T$ is a "total differential".
A simple example may illustrate this:
Let

$$F(x, y) = x^2 y \quad , \tag{3.24}$$

thus

$$dF(x, y) = 2xydx + x^2 dy \quad . \tag{3.25}$$

We search for solutions $F(x, y) = const.$, but only know the total differential

$$0 = dF = 2xydx + x^2 dy \quad \text{, and after transformation} \quad dy/dx = -2y/x \quad . \tag{3.26}$$

The four equations are equivalent to some extent, but we lost a factor x in the last equation.
For the solution $0 = 2ydx + xdy$ it is hard (impossible) to find a function with $dG(x, y) = 2ydx + xdy$ (try?!?). We first have to multiply with the factor x.
Same as in the above example only after multiplying with the integrating factor $1/T$ a total differential is found

$$dS = \delta Q/T \quad . \tag{3.27}$$