1.5 The inverse temperature as an integrating factor

The phrase "integrating factor" origins from the theory for solving differential equations. The factor $1/T$ is called an integrating factor for δQ, since $dS = \delta Q/T$ is a "total differential".
A simple example may illustrate this:
Let

$$F(x,y) = x^2y \quad , \tag{1.6}$$

thus

$$dF(x,y) = 2xydx + x^2dy \quad . \tag{1.7}$$

We search for solutions $F(x,y) = const.$, but only know the deviation

$$0 = dF = 2xydx + x^2dy \quad , \tag{1.8}$$

and after transformation

$$dy/dx = -2y/x \quad . \tag{1.9}$$

The four equations are equivalent to some extend, but we lost the factor $1/x$ in the last equation.
For the solution $0 = 2ydx + xdy$ it is hard (impossible) to find a function with $dG(x,y) = 2ydx + xdy$ (try?!?). We first have to multiply with the factor x.
Same as in the above example only after multiplying with the integrating factor $1/T$ a total differential is found

$$dS = \delta Q/T \quad . \tag{1.10}$$