4.1 Einstein's interpretation of the Bose statistics

The spectral density of radiation
We will discuss the electromagnetic radiation coming out of a black radiator:
For photons the dispersion relation holds:

$$\omega = ck \tag{4.1}$$

with c: velocity of light.
Just as for the Debye model we find

$$N(k) = 2\left(\frac{L}{2\pi}\right)^3 \frac{4}{3}\pi k^3 \tag{4.2}$$

The factor 2 sums up both planes for transverse electromagnetic waves. Again we get

$$D(\omega) = \left(\frac{Vk^2}{\pi^2}\right)\frac{dk}{d\omega} = \frac{V\omega^2}{\pi^2 c^3} \qquad , \tag{4.3}$$

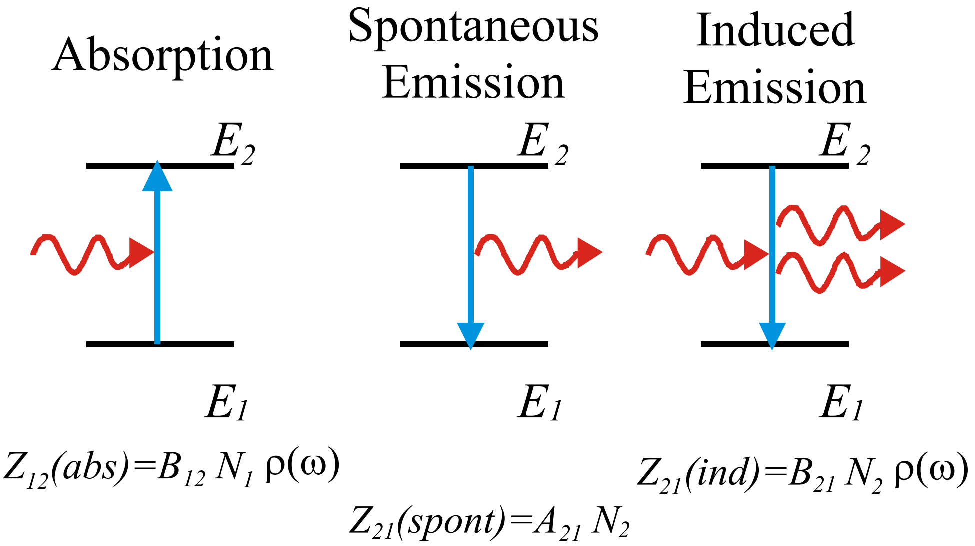
and consequently the spatial spectral density of radiation

$$\rho(\omega, T) = \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \qquad . \tag{4.4}$$

This is the famous radiation law of Max Planck.

Planck's radiation law as a balance between absorption and emission

What kinds of radiation interaction exist between two energy levels E_1 and E_2?



1. The rate of absorption processes per time is proportional to the number N_1 of atoms in the ground state and the energy density $\varrho(\omega)$ of the electromagnetic field at the energy ω is

   $$Z_{12}^{(abs)} = B_{12} N_1 \varrho(\omega) \qquad . \tag{4.5}$$

   B_{12} is called the Einstein coefficient for absorption. This equation holds not only for thermodynamic equilibrium; therefor the parameter T was omitted.

2. The number of spontaneous emission processes per time is proportional to the number N_2 of atoms in an excited state

   $$Z_{21}^{(spont)} = A_{21} N_2 \qquad . \tag{4.6}$$

   If this process is dominant, we find

   $$Z_{21}^{(spont)} = -\frac{dN_2}{dt} \qquad , \tag{4.7}$$

i.e.

$$\frac{dN_2}{dt} = -A_{21}N_2 \qquad , \tag{4.8}$$

with the solution

$$N_2(t) = N_2(0)\exp\left(-A_{21}t\right) = N_2(0)\exp\left(-\frac{t}{\tau}\right) \qquad , \tag{4.9}$$

i.e. the excited atoms relax exponentially with a mean lifetime τ defined by

$$A_{21} = \frac{1}{\tau} \qquad . \tag{4.10}$$

A_{12} is called spontaneous transition probability.
τ is called relaxation time. It quantifies how fast a system reaches again equilibrium after a perturbation.

3. Under the influence of radiation we find "induced" transitions. The number of transitions per time interval is proportional to the number N_2 of excited atoms and the energy density $\varrho(\omega)$ of the radiation

$$Z_{21}^{(ind)} = B_{21}N_2\varrho(\omega) \qquad . \tag{4.11}$$

B_{21} is called Einstein coefficient of "induced" ("stimulated") emission.

For steady state (constant occupation numbers N_1 and N_2) the following relation must hold:

$$Z_{12}^{(abs)} = Z_{21}^{(spont)} + Z_{21}^{(ind)} \qquad , \tag{4.12}$$

i.e.

$$B_{12}N_1\varrho(\omega) = A_{21}N_2 + B_{21}N_2\varrho(\omega) \qquad , \tag{4.13}$$

and consequently

$$\varrho(\omega) = \frac{A_{21}}{\frac{N_1}{N_2}B_{12} - B_{21}} = \frac{\frac{A_{21}}{B_{21}}}{\frac{N_1}{N_2}\frac{B_{12}}{B_{21}} - 1} \qquad . \tag{4.14}$$

For thermodynamic equilibrium at a temperature T we find:
(canonical ensemble)

$$\frac{N_1}{N_2} = \exp\left(\frac{\epsilon_2 - \epsilon_1}{kT}\right) = \exp\left(\frac{\hbar\omega}{kT}\right) \qquad . \tag{4.15}$$

leading to

$$\varrho(\omega, T) = \frac{\frac{A_{21}}{B_{21}}}{\frac{B_{12}}{B_{21}}\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \qquad . \tag{4.16}$$

Comparing with the Bose statistics we find:

$$A_{21} = \frac{\hbar\omega^3}{\pi^2c^3}B_{21} \qquad , \tag{4.17}$$

and

$$B_{12} = B_{21} \qquad . \tag{4.18}$$

- The above discussion is not a deduction of the Bose statistics but an interpretation of the underlying processes.
- Essential for the interpretation are not only the spontaneous emission and absorption but also the stimulated emission.
- These three processes are not restricted to phonons and black body radiator but belong to Bosons in general.
- Non equilibrium phenomena are described as well by these processes.
- The spontaneous emission leads to a random photon radiation. Therefor no phase coupling exists between the photons.
- The induced emission generates light with exactly the same phase as the incoming light. We get a coherent light beam (in time and in space, interference's are possible, light is parallel,...)

LASER light origins from this induced emission process!