3.9 Specific heat capacity of the free electron gas (Fermions)

We again apply the Eq. (3.7) to (3.9) for the calculation of the particle number and the energy:

$$N(T,V,\mu) = \int_{-\infty}^{+\infty} D(\epsilon) f_{\beta\mu}(\epsilon) d\epsilon \tag{3.46}$$

and

$$E(T,V,\mu) = \int_{-\infty}^{+\infty} \epsilon D(\epsilon) f_{\beta\mu}(\epsilon) d\epsilon \qquad . \tag{3.47}$$

$f_{\beta\mu}(\epsilon)$ is the Fermi statistics; obviously we find:

$$N(T,V,\mu) = \int_{-\infty}^{+\infty} D(\epsilon) f_{\beta\mu}(\epsilon) d\epsilon = \int_{-\infty}^{\epsilon_F} D(\epsilon) d\epsilon \tag{3.48}$$

Multiplying both sides of Eq. (3.48) with ϵ_F we get

$$\left(\int_{-\infty}^{\epsilon_F} + \int_{\epsilon_F}^{+\infty} \right) \epsilon_F D(\epsilon) f_{\beta\mu}(\epsilon) d\epsilon = \int_{-\infty}^{\epsilon_F} \epsilon_F D(\epsilon) d\epsilon \tag{3.49}$$

Since we calculate the derivation with respect to temperature, we can subtract a constant from the inner energy. We get

$$\Delta U = \int_{-\infty}^{+\infty} \epsilon D(\epsilon) f_{\beta\mu}(\epsilon) d\epsilon - \int_{-\infty}^{\epsilon_F} \epsilon_F D(\epsilon) d\epsilon \tag{3.50}$$

Combining the equations (3.49) and (3.50) we find

$$\Delta U = \int_{\epsilon_F}^{+\infty} (\epsilon - \epsilon_F) D(\epsilon) f_{\beta\mu}(\epsilon) d\epsilon - \int_{-\infty}^{\epsilon_F} (\epsilon - \epsilon_F) \left(1 - f_{\beta\mu}(\epsilon)\right) D(\epsilon) d\epsilon \qquad . \tag{3.51}$$

The first term describes the excitation of an electron from the energy ϵ_F to ϵ and the second term the excitation from ϵ to ϵ_F.
Finally we get

$$c = \frac{d\Delta U}{dT} = \int_{-\infty}^{+\infty} (\epsilon - \epsilon_F) D(\epsilon) \frac{\partial f_{\beta\mu}(\epsilon)}{\partial T} d\epsilon \qquad . \tag{3.52}$$

Only around the Fermi energy $\frac{\partial f_{\beta\mu}(\epsilon)}{\partial T}$ differs from zero; we therefor substitute $D(\epsilon)$ by $D(\epsilon_F)$ and take

$$x := \frac{\epsilon - \epsilon_F}{kT} \qquad , \tag{3.53}$$

leading to

$$c \approx k^2 T D(\epsilon_F) \int_{-\infty}^{+\infty} \frac{x^2 e^x}{(e^x+1)^2} dx = \frac{\pi^2}{3} D(\epsilon_F) k^2 T \qquad , \tag{3.54}$$

Since for free electrons

$$N(\epsilon) = const. * \epsilon^{\frac{3}{2}} \qquad , \tag{3.55}$$

we get

$$\frac{\partial N}{\partial \epsilon} = const. * \frac{3}{2} \epsilon^{\frac{1}{2}} = \frac{3}{2} \frac{N}{\epsilon} \qquad . \tag{3.56}$$

Therefore

$$D(\epsilon_F) = \frac{3N}{2\epsilon_F} = \frac{3N}{2kT_F} \qquad , \tag{3.57}$$

and

$$c = \frac{\pi^2}{2} Nk \frac{T}{T_F} \qquad . \tag{3.58}$$

For room temperature and a typical Fermi temperature of several 1000^oK follows

$$c \approx \frac{1}{100} Nk \qquad . \tag{3.59}$$

Thus at room temperature the heat capacity of electrons is not important.