3.7 The Debye Model

Calculating the number of particles for a linear dispersion relation we get from Eq (3.32) a limiting frequency

$$\omega_D^3 = 6\pi^2 v^3 \frac{N}{V} \qquad . \tag{3.35}$$

The corresponding density of states is

$$D(\omega) = \frac{V\omega^2}{2\pi^2 v^3} \qquad . \tag{3.36}$$

Taking into account the three orientation in space we get for the inner energy

$$U = 3 \int d\omega \frac{V\omega^2}{2\pi^2 v^3} \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \qquad . \tag{3.37}$$

leading to:

$$C_V = \frac{dU}{dT} = 9Nk \left(\frac{T}{\Theta}\right)^3 \int_0^{x_D} dx \frac{x^4 e^x}{(e^x - 1)^2} \qquad . \tag{3.38}$$

with

$$x_D := \frac{\hbar\omega_D}{kT} := \frac{\Theta}{T} \qquad . \tag{3.39}$$

and Θ: Debye temperature.
The limiting cases are:

I: $T \ll \Theta$, i.e. $x_D \to \infty$

$$C_V = 9Nk \left(\frac{T}{\Theta}\right)^3 \frac{\pi^4}{15} \tag{3.40}$$

II: $T \gg \Theta$, i.e. $x_D \to 0$

$$\frac{x^4 e^x}{(e^x - 1)^2} \approx x^2 \tag{3.41}$$

and consequently

$$C_V = 9Nk \left(\frac{T}{\Theta}\right)^3 \int_0^{x_D} x^2 dx = 3Nk \tag{3.42}$$

This is the expected classical result (the Hamiltonian is a bilinear function of the coordinates).