1.4 Eigenvalues and Eigenvectors of matrices

In general the transformation A of the vectors x leads to

$$y = Ax \tag{1.20}$$

where y is rotated against x and the lengths of both vectors differ.
Now we are looking for special vectors v with

$$Av = \lambda v = \lambda E v \tag{1.21}$$

These "Eigenvectors" only change their length, when the operator A is applied to them.
Eq. (1.21) can be rewritten as

$$(A - \lambda E)v = 0 \tag{1.22}$$

This equations has only nontrivial solutions if

$$\det(A - \lambda E) = 0 \tag{1.23}$$

The zeros of this characteristic polynom are the Eigenvalues λ_i of the operator A.
For each Eigenvalue λ_i we can calculate the Eigenvectors v_i by solving the equation

$$(A - \lambda_i E)v_i = 0 \tag{1.24}$$

Example:

The matrix $\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ has the characteristic polynom $\lambda^2 - 6\lambda + 8 = 0$ with its solutions $\lambda_1 = 2$ and $\lambda_2 = 4$.
The corresponding Eigenvectors are $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
The adjacent unitary transformation is therefore $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.
An illustrative example for an Eigenvalue problem you can find in the math-script.

Eigenvalues of linear Hermitian Operators are always real

Let A be a Hermitian operator with the Eigenfunction f_1 and the Eigenvalue a_1; it follows that

$$a_1\langle f_1|f_1\rangle = \langle f_1|Af_1\rangle = \langle Af_1|f_1\rangle = a_1^*\langle f_1|f_1\rangle \tag{1.25}$$

thus a_1 is real.

Eigenvectors belonging to different Eigenvalues are orthogonal

Let A be a Hermitian operator with Eigenfunctions f_1 and f_2. Let $a_1 \neq a_2$ be the corresponding Eigenvalues. We find

$$a_2\langle f_1|f_2\rangle = \langle f_1|Af_2\rangle = \langle Af_1|f_2\rangle = a_1^*\langle f_1|f_2\rangle \tag{1.26}$$

Consequently

$$(a_2 - a_1^*)\langle f_1|f_2\rangle = 0 \qquad ; \tag{1.27}$$

since $(a_2 - a_1^*) \neq 0$ we find that f_1 and f_2 are orthogonal.

Degenerated Eigenvalues

Let A be a Hermitian operator with the Eigenfunctions f_1 and f_2. Let $a_1 = a_2 = a$ be the corresponding Eigenvalue, i.e.

$$\begin{aligned} A|f_1\rangle &= a|f_1\rangle \\ A|f_2\rangle &= a|f_2\rangle \end{aligned} \tag{1.28}$$

We find:

$$A\left(x|f_1\rangle + y|f_2\rangle\right) = xa|f_1\rangle + ya|f_2\rangle = a\left(x|f_1\rangle + y|f_2\rangle\right) \qquad . \tag{1.29}$$

Thus degenerated Eigenfunctions form a sub vector space. This sub space is perpendicular to all other Eigenvectors. The orthonormalization procedure of Schmidt allows to find a set of orthogonal vectors of length 1 in this subspace.

Orthonormal base of vectors

The system of Eigenvectors for every Hermitian matrix can thus be transformed to a set of orthonormal vectors e_i, with represent a basis in the vector space:

$$\langle e_i | e_j \rangle = \delta_{i,j} \qquad . \tag{1.30}$$

Each vector can be written as

$$\langle a| = \sum_i \langle a|e_i\rangle\langle e_i| \qquad . \tag{1.31}$$

which is called the Closure-relation. The $a_i = \langle a|e_i\rangle$ are called the components of the vector for the basis e_i. In component representation the vector is just written as $\langle a| = (a_1, a_2, a_3, ..., a_n)$.

REMARK: Remember the Bra-, Ket- representation of vectors; for complex vectors the components have to be chosen complex conjugated when changing from Bra- to Ket- representation or vice versa.