1.3 The linear Hermitian operator

A linear operator A is called Hermitian, if:

$$\langle x|Ay\rangle = \langle Ax|y\rangle = \langle x|A|y\rangle \qquad \text{(Bra ..... Ket)} \tag{1.11}$$

The third notation is used to point out the symmetry of the first equality.

Examples:

Hermitian matrices:

Applying the above definition to a matrix, we find:

$$|Ab\rangle_i = \sum_j A_{i,j} b_j \qquad \text{and} \qquad |Aa\rangle_i = \sum_j A_{i,j} a_j \tag{1.12}$$

With

$$\langle Aa|_i = \sum_j A^*_{i,j} a^*_j \tag{1.13}$$

follows

$$\langle a|Ab\rangle = \sum_{i,j} a^*_i A_{i,j} b_j \qquad \text{and} \qquad \langle Aa|b\rangle = \sum_{i,j} a^*_j A^*_{i,j} b_j \tag{1.14}$$

For Hermitian matrices we find according to Eq. (1.14):

$$A_{i,j} = A^*_{j,i} \tag{1.15}$$

Multiplication with x

In this case

$$\langle f|Ag\rangle = \langle Af|g\rangle = \langle f|A|g\rangle \tag{1.16}$$

is written as

$$\int_{-\infty}^{+\infty} f^*(x) x g(x) dx = \int_{-\infty}^{+\infty} x^* f^*(x) g(x) dx \tag{1.17}$$

Since x is a real value, the multiplication with x is a Hermitian operator.
The differential operator

As proofed by the following calculation, the differential operator is not Hermitian. We thus choose the correct operator

$$Af = i\frac{d}{dx} f \qquad i\text{: imaginary unit} \tag{1.18}$$

We find:

$$\begin{aligned} &\langle f|Ag\rangle = \\ &\int_{-\infty}^{+\infty} f^*(x) i \frac{dg}{dx}(x) dx = i f^*(x) g(x)\Big|_{-\infty}^{+\infty} - i \int_{-\infty}^{+\infty} \frac{df^*}{dx}(x) g(x) dx \\ &= \int_{-\infty}^{+\infty} i^* \frac{df^*}{dx}(x) g(x) dx \\ &= \langle Af|g\rangle \end{aligned} \tag{1.19}$$

The partial integration gives a "boundary term". This term is zero, because f and g must be square integrable and thus are zero for large and "small" x-values.
The i leading the differential operator compensates for the minus sign which occurs in the second term after partial integration.