1.2 Mathematical Basics

The Vector Space

Let $(G, +)$ be a commutative group and $a, b \in G$. Let $(K, +, *)$ be a field and $\alpha, \beta \in K$. V is a vector space, if

$$\begin{aligned} \alpha(a+b) &= \alpha a + \alpha b \\ (\alpha+\beta)a &= \alpha a + \beta a \\ (\alpha\beta)a &= \alpha(\beta a) \\ 1a &= a \end{aligned} \tag{1.2}$$

Examples:

n-dim. real/complex space	real functions

Linear functions / Linear operators

Let V be a vector space. A is a linear function, if:

$$\begin{aligned} &A : V \to V, f_1, f_2 \in V, \lambda \in \mathbb{R}\ (\mathbb{C}) \\ &A(f_1 + f_2) = A(f_1) + A(f_2) \\ &A(\lambda f_1) = \lambda A(f_1) \end{aligned} \tag{1.3}$$

Examples:

matrix M, vectors $\vec{a}, \vec{b}$	operator A, function f
$M(\vec{a} + \vec{b}) = M(\vec{a})) + M(\vec{b})$	a) $Af = af$ (multiplication with a constant)
	b) $Af = xf$ (multiplication with x)
	c) $Af = \frac{df}{dx}$ (differentiation)
	d) $Af = \int K(x-y)f(y)dy$ (folding with an integral kernel)

Inner product or scalar product

Let V be a vector space over the field K (real or complex space), $\langle , \rangle : V \times V \to K$ is called scalar product, if:

$$\begin{aligned} \langle x, y\rangle &= \langle y, x\rangle^* \\ \langle x_1 + x_2, y\rangle &= \langle x_1, y\rangle + \langle x_2, y\rangle \\ \langle x, \alpha y\rangle &= \alpha\langle x, y\rangle \\ \text{i.e.} \qquad \langle \alpha x, y\rangle &= \alpha^*\langle x, y\rangle \\ \langle x, x\rangle &\in \mathbb{R}^+ \qquad \text{for} \qquad x \neq 0 \end{aligned} \tag{1.4}$$

Examples:

$\langle a, b\rangle = \sum_i a_i^* b_i$	$\langle f, g\rangle = \int f^*(x)g(x)dx$ REMARK: f and g should be square integrable

A vector space with a scalar product is call Prehilbert space. A complete Prehilbert space is called Hilbert space.
Mathematically: *Quantization* occurs, because the space of square integrable functions is not complete.
Using the scalar product, some very important properties of vectors can be proofed:

The norm of a vector

is defined by

$$|x| = \sqrt{\langle x, x\rangle} \tag{1.5}$$

Examples:

$\lvert a\rvert = \sqrt{\sum_{i=1}^{n} \lvert a_i\rvert^2}$	$\lvert f\rvert = \sqrt{\int \lvert f(x)\rvert^2 \, dx}$

Two vectors are orthogonal (perpendicular):

$$\langle x, y\rangle = 0 \tag{1.6}$$

Two vectors are parallel:

$$x = \lambda y \qquad \lambda \in \mathbb{C} \tag{1.7}$$

The Schwarz inequality

$$\lvert\langle f, g\rangle\rvert \leq \lvert f\rvert\lvert g\rvert \tag{1.8}$$

is a direct consequence of the definition of the scalar product.

The Triangle inequality

$$\lvert f + g\rvert \leq \lvert f\rvert + \lvert g\rvert \tag{1.9}$$

again is a direct consequence of the definition of the scalar product. It can be transformed into

$$\lvert f - g\rvert \geq \lvert f\rvert - \lvert g\rvert \tag{1.10}$$