2.13.3 Calculation of Eigenvectors

Example:

$$\tilde{A} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \qquad \text{eigenvalues:} \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = -1 \end{matrix}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 3 \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{matrix} \alpha_1 - 2\alpha_2 = 3\alpha_1 & \Rightarrow & \alpha_1 + \alpha_2 = 0 \\ -2\alpha_1 + \alpha_2 = 3\alpha_2 & \Rightarrow & \alpha_1 + \alpha_2 = 0 \end{matrix}\right\} \alpha_1 + \alpha_2 = 0 \text{ or } \alpha_1 = -\alpha_2$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \alpha_1\text{-arbitrary} \rightarrow$$

Eigenvectors determined up to an arbitrary factor.

e.g. $\alpha_1 = 1 \quad \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an Eigenvector of $\tilde{A}$

$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = -1 \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rightarrow \begin{matrix} \alpha_1 - 2\alpha_2 = -\alpha_1 \\ \\ -2\alpha_1 + \alpha_2 = \alpha_2 \end{matrix}$$

$$\rightarrow \quad \left.\begin{matrix} \alpha_1 - \alpha_2 = 0 \\ \alpha_1 - \alpha_2 = 0 \end{matrix}\right\} \rightarrow \alpha_1 = \alpha_2$$

$\Rightarrow \vec{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ other linear independent EW

$\rightarrow \vec{x}_1 = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ \vec{x}_2 = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are two linear independent Eigenvectors of the matrix $\tilde{A}$

General for calculation of Eigenvector of a $N \times N$ matrix $\tilde{A}$

step 1: $\det(\tilde{A} - \lambda\tilde{I}) = 0 \rightarrow$ characteristic polynomial in λ $P(\lambda)$

step 2: find the N solutions of $P(\lambda) = 0$

step 3: choose $\vec{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$ with N unknown α_j and solve the homogeneous system $(\tilde{A} - \lambda\tilde{I})\vec{x} = 0$ for these α_j.

- if no ,multiple zeros of $P(\lambda)$ then $\alpha_1, \ldots, \alpha_{N-1}$ can be expressed with one free parameter, say α_N
- if multiple zeros are present, say if λ_0 is a k-times zero of $P(\lambda)$ then $\alpha_1, \ldots, \alpha_{n-k}$ can be calculated with k free parameters, say $\alpha_{N-k+1}, \ldots, \alpha_N$.

Example:

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} P(\lambda) = (1-\lambda)^2\lambda(\lambda - 2) \\ \\ EW : \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 2 \end{matrix}$$

Eigenvectors:

$$\vec{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \rightarrow \left(\tilde{A}\right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 1 \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} \alpha_1 + \alpha_4 & = & \alpha_1 \\ \alpha_2 & = & \alpha_2 \\ \alpha_3 & = & \alpha_3 \\ \alpha_1 + \alpha_4 & = & \alpha_4 \end{matrix} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \text{arbitrary} \quad \Bigg| \quad \alpha_4 = \alpha_1 = 0$$

$$\rightarrow \quad \vec{x} = \begin{pmatrix} 0 \\ \alpha_2 \\ \alpha_3 \\ 0 \end{pmatrix} \text{ is an Eigenvector with } \alpha_2, \alpha_3 \text{ arbitrary! } N=4,\, k=2 \rightarrow \begin{array}{l} \alpha_1, \alpha_4 \text{ fixed} \\ \alpha_2, \alpha_3 \text{ arbitrary} \end{array}$$

$$\rightarrow \quad \left(\tilde{A}\right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \Rightarrow \begin{array}{r} \alpha_1 + \alpha_4 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \\ \alpha_1 + \alpha_4 = 0 \end{array} \quad \alpha_4 = -\alpha_1$$

$$\rightarrow \vec{x} = \begin{pmatrix} \alpha_1 \\ 0 \\ 0 \\ -\alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \alpha_1, \text{ arbitrary}$$

$$\left(\tilde{A}\right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \rightarrow \begin{array}{rcl} \alpha_1 + \alpha_4 & = & 2\alpha_1 \\ \alpha_2 & = & 2\alpha_2 \rightarrow \alpha_2 = 0 \\ & & \\ \alpha_3 & = & 2\alpha_3 \rightarrow \alpha_3 = 0 \\ \alpha_1 + \alpha_4 & = & 2\alpha_4 \end{array} \rightarrow \alpha_4 = \alpha_1$$

$$\rightarrow \vec{x} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Summary:

$$\lambda_1 = 1 \quad (2 \text{ times}) \quad , \quad \vec{x} = \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \\ \alpha_3 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 0 \quad , \quad \vec{x} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_3 = 2 \quad , \quad \vec{x} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus:

$$\left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

is a set of 4 lines independent Eigenvectors of $\tilde{A}$. The factor $\frac{1}{\sqrt{2}}$ is because we want $|\vec{x}| = 1$.
in general: if $\tilde{A}$ $N \times N$ then there are N linear independent Eigenvectors, which may be difficult to find!
Diagonal matrices:

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_N \end{pmatrix} \text{ than } P(\lambda) = (\alpha_1 - \lambda) \cdot (\alpha_2 - \lambda) \cdot \ldots \cdot (\alpha_N - \lambda). \\ P(\lambda) &= 0 \rightarrow \lambda_1 = \alpha_1, \ldots, \lambda_N = \alpha_N \\ \text{thus: } \det(\tilde{A}) &= \lambda_1 \cdot \ldots \cdot \lambda_N = \Pi_{j=1}^N \lambda_j \end{aligned}$$

$\Rightarrow$ we will see that this can be achieved by a transformation for certain matrices ($\Rightarrow$ exercises!)