2.13.2 Example: 2x2 and 4x4 matrix

(i)

$$\tilde{A} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \quad 2 \times 2$$

$$\begin{aligned}
\det(\tilde{A} - \lambda\tilde{I}) &= \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0 \\
\det(\tilde{A} - \lambda\tilde{I}) &= P(\lambda) = \lambda^2 - 2\lambda - 3 = 0
\end{aligned}$$

$$\begin{aligned}
P(\lambda) &= \lambda^2 - 2\lambda - 3 \text{ is the characteristic polynomial associated with the matrix } \tilde{A} \quad . \\
P(\lambda) &= 0 \;\rightarrow\; \lambda_{1/2} = 1 \pm \sqrt{1^2+3} = 1 \pm 2 \rightarrow \begin{array}{c} \lambda_1 = 3 \\ \lambda_2 = -1 \end{array} \\
&\Rightarrow \quad \text{Eigenvalues of matrix } \tilde{A} \text{ are } \lambda_1 = 3 \text{ and } \lambda_2 = -1
\end{aligned}$$

(ii)

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad 4 \times 4$$

$$\begin{aligned}
\det(\tilde{A} - \lambda\tilde{I}) &= \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 \left[(1-\lambda)^2 - 1\right] = (1-\lambda)^2 \left(\lambda^2 - 2\lambda + 1 - 1\right) \\
\Rightarrow P(\lambda) &= (1-\lambda)^2 \lambda(\lambda - 2) \;\; \text{(important: not further simplifying!!)}
\end{aligned}$$

Eigenvalues:

$$P(\lambda) = 0 \quad \rightarrow \quad \begin{array}{l} \lambda_1 = 1 \quad (\rightarrow \text{ two times because of } (1-\lambda)^2) \\ \lambda_2 = 0 \\ \lambda_3 = 2 \end{array}$$

Matrix $\tilde{A}$, $N \times N$ $\Rightarrow$ Polynomial $P(\lambda)$ is of degree N; N Eigenvalues exist, since $P(\lambda) = 0$ has N solutions
However, example (ii) only yields 3 EW for a 4×4 matrix because $\lambda_1 = 1$ is a multiple zero.
in general λ_0 is called an j-times zero (zero of order j) of $P(\lambda)$ if

$$P(\lambda) = (\lambda - \lambda_0)^j P_0(\lambda) \quad \text{and} \quad P_0(\lambda) \text{ is of degree } N - j$$

Example:

(i)

$$P(\lambda) = (\lambda - 1)^2 \lambda(\lambda - 2)$$

$$\left.\begin{array}{lcl} \lambda_1 = 1 & \rightarrow & 2 \text{ times} \\ \lambda_2 = 0 & \rightarrow & 1 \text{ times} \\ \lambda_3 = 2 & \rightarrow & 1 \text{ times} \end{array}\right\} \text{ 4 zeros since } \lambda_1 \text{ counts twice!}$$

(ii)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \quad P(\lambda) = (1-\lambda)^4 = (\lambda - 1)^4$$

$$\rightarrow \quad \lambda_1 = 1 \text{ is a 4 times zero of } P(\lambda)$$