2.10.1 Examples: Using cofactors for matrix inversion

(i)

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \tilde{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^{\top} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(ii)

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad \tilde{A}^{-1} = \ ?$$

$$\det(\tilde{A}) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = + \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2$$

sub minor determinants:

$$A_{11} = +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{12} = -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2 \qquad A_{13} = +\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$A_{21} = -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0 \qquad A_{22} = +\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2 \qquad A_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$A_{31} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = +2 \qquad A_{33} = +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$\Rightarrow \tilde{A}^{-1} = \frac{1}{-2}\begin{pmatrix} +1 & -2 & -3 \\ 0 & -2 & 0 \\ -1 & +2 & 1 \end{pmatrix}^{\top} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ +1 & 1 & -1 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

Test:

$$\tilde{A}\tilde{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(o.k.)}$$

If $\tilde{A}$ is an $N \times N$ matrix with $\det(\tilde{A}) \neq 0$ then there exists an inverse with $\tilde{A}\tilde{A}^{-1} = \tilde{A}^{-1}\tilde{A} = \tilde{I}$ and vice versa. If $\tilde{A}^{-1}$ exists, then $\det(\tilde{A}) \neq 0$. Thus $\det(\tilde{A}^{-1}) = \frac{1}{\det(\tilde{A})}$ because $\det(\tilde{A}\tilde{A}^{-1}) = 1 = \det(\tilde{I})$