4.8.1 Examples for finding extrema

(i)

$$\begin{aligned} f(x,y) &= x^2+y^2 \\ \to \hat{H}(x,y) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \to \text{pos. definite, i.e. minimum} \end{aligned}$$

(ii)

$$\begin{aligned} f(x,y) &= -x^2-y^2 \\ \to \hat{H}(x,y) &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \to \text{neg. definite, i.e. maximum} \end{aligned}$$

(iii)

$$\begin{aligned} f(x,y) &= x^2-y^2 \\ \to \hat{H}(x,y) &= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \to \text{indefinite, i.e. no extremum, saddle point} \end{aligned}$$

(iv) semi-cases difficult:

$$f(x,y) = x^2+y^4 \to \vec{\nabla} f = \begin{pmatrix} 2x \\ 4y^3 \end{pmatrix} \to \hat{H}(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

$\hat{H}$ pos. semi-definite?$\to$minimum

$$f(x,y) = x^2 \to \vec{\nabla} f = \begin{pmatrix} 2x \\ 0 \end{pmatrix} \to \hat{H}(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\hat{H}$ pos. semi-definite?$\to$mall values $x=0$ y-arbitrary are minima

$$f(x,y) = x^2+y^3 \to \vec{\nabla} f = \begin{pmatrix} 2x \\ 3y^2 \end{pmatrix} \to \hat{H}(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 6y \end{pmatrix}$$

$\hat{H}(0,0)$ pos. semi-definite?$\to$but no extremum, complicated saddle point!

(v)

$$f(x_1,x_2) = e^{-(x_1^2+x_2^2-4x_1x_2-2x_1)} \quad \vec{\nabla} f = e^{-(x_1^2+x_2^2-4x_1x_2-2x_1)} \begin{pmatrix} -2x_1+4x_2+2 \\ -2x_2+4x_1 \end{pmatrix}$$

$$\vec{\nabla} f = -2e^{-(x_1^2+x_2^2-4x_1x_2-2x_1)} \begin{pmatrix} 2x_2-x_1-1 \\ 2x_1-x_2 \end{pmatrix} \quad \begin{aligned} \to\ 2x_2-x_1 &= 1 \\ \to\ 2x_1-x_2 &= 0 \\ x_2 &= 2x_1 \end{aligned} \quad \to \quad \begin{aligned} 4x_1-x_1 &= -1 \\ x_1 &= -\tfrac{1}{3} \\ x_2 &= -\tfrac{2}{3} \end{aligned}$$

$$\begin{aligned} \tilde{H}(x_1,x_2) &= 2e^{-(x_1^2+x_2^2-5x_1x_2-2x_1)} * \\ & \begin{pmatrix} -1+2(2x_2-x_1+1)^2 & 2+2(2x_1-x_2)(2x_2-x_1+1) \\ 2+2(2x_1-x_2)(2x_2-x_1+1) & -1+2(2x_1-x_2)^2 \end{pmatrix} \end{aligned}$$

Ignoring the positive prefactor we find

$$\tilde{H}(-\frac{1}{3},-\frac{2}{3}) \propto \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

i.e. the Hesse matrix is negative-definite implicating that the extremum at $(-\frac{1}{3},-\frac{2}{3})$ is a maximum.