4.5.1 Examples: For Jacobi matrix and Jacobi determinant

(i)

$$\begin{aligned}\vec{f}(x_1,x_2) &= \begin{pmatrix} x_1+x_2^2 \\ x_1^2 \end{pmatrix}\\ \to \tilde{A}(x_1,x_2) &= \begin{pmatrix} 1 & 2x_2a \\ 2x_1 & 0 \end{pmatrix} \quad \text{as before!}\end{aligned}$$

(ii)

$$\begin{aligned}\vec{f}(r,\vartheta,\varphi) &= \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix} = \begin{pmatrix} f_1(r,\vartheta,\varphi) \\ f_2(r,\vartheta,\varphi) \\ f_3(r,\vartheta,\varphi) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \hat{=} \text{ spherical coordinates!}\\ \tilde{A} &= \begin{pmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix}\\ \det\tilde{A} &= r^2\sin\vartheta \quad \text{(volume!)}\end{aligned}$$

(iii)

$$\begin{aligned}\vec{f}(r,\varphi,z) &= \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \hat{=} \text{ cylindrical coordinates}\\ \tilde{A} &= \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\\ \det\tilde{A} &= r \quad \text{(volume!)}\end{aligned}$$

(iv)

$$\begin{aligned}f:\mathbb{R}^2 &\to \mathbb{R} \qquad f(x_1,x_2) = x_1^2+x_2^3\\ \vec{A} &= \vec{\nabla}f = \begin{pmatrix} 2x_1 \\ 3x_2^2 \end{pmatrix} \qquad \vec{\nabla}f:\mathbb{R}^2\to\mathbb{R}^2\end{aligned}$$

second derivative is now a matrix:

$$\tilde{D}_2 f = \begin{pmatrix} 2 & 0 \\ 0 & 6x_2 \end{pmatrix} \leftarrow M=1,\ \text{Hesse}(\vec{\nabla}f)^\top$$

(v)

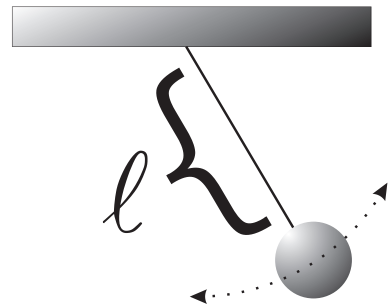
$$\begin{aligned}f &: \mathbb{R}^N \to \mathbb{R}\\ f(\vec{x}) &= |\vec{x}|^2 = x_1^2+x_2^2+\ldots+x_N^2\\ \vec{A} &= \vec{\nabla}f = \begin{pmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_N \end{pmatrix} = 2\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = 2\vec{x}\end{aligned}$$

second derivative:

$$\tilde{D}_2 f = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} = 2\tilde{I}$$

(vi) physics example: g-measurement with pendulum

$$\text{period } T = 2\pi\sqrt{\frac{l}{g}} \to g = \frac{4\pi^2 l}{T^2} = g(l,T)$$



Measurement: $l \pm \Delta l$, $T \pm \Delta T$; $\Delta l, \Delta T$ errors of measurement.
($l = 1$m, $\Delta l = 0.001$, $T = 2$s, ΔT =0.1s)
What is the error Δg of g?
Total differential:

$$\begin{aligned}
\Delta g &= |\frac{\partial g}{\partial l}|\Delta l + |\frac{\partial g}{\partial T}|\Delta T \\
&= \left|\frac{4\pi^2}{T^2}\right|\Delta l + \left|\frac{4\pi^2 l}{T^3}(-2)\right|\Delta T \\
&= 0.01\frac{m}{s^2} + 1\frac{m}{s^2} = 1.01\frac{m}{s^2}
\end{aligned}$$