3.15.1 Convolution Theorem: Proof and example

Let F and G be the Fourier transforms of f and g, i.e.

$$\begin{aligned} F(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ipx} dx \\ G(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) e^{-ipy} dy \end{aligned}$$

Then

$$\mathcal{F}^{-1}\left\{F(p)\cdot G(p)\right\}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ipx} dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) e^{-ipy}\, dy \right) e^{ipz} dp$$

Since

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ip(x+y-z)} dp = \delta(x+y-z)$$

We finally get

$$\begin{aligned} \mathcal{F}^{-1}\left\{F(p)\cdot G(p)\right\}(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x) g(y) \delta(x+y-z) dx dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) g(z-x) dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ f \star g \right\}(z) \end{aligned}$$

Example

$$\begin{aligned} F(p) &= \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+p^2} \qquad f(x) = e^{-\alpha|x|} \\ G(p) &= \frac{2aA}{\sqrt{2\pi}} \frac{\sin(pa)}{pa} \qquad g(x) = \begin{cases} A & |x| < a \\ 0 & |x| > a \end{cases} \end{aligned}$$

$$\begin{aligned} \rightarrow \mathcal{F}\left\{\sqrt{2\pi} F(p) G(p)\right\} &= \int_{-\infty}^{+\infty} e^{-\alpha|x-t|} A dt = A \overbrace{\int_{-a}^{+a} e^{-\alpha|x-t|} dt}^{\text{different cases}} \\ &= h(x) = e^{-\alpha|x|} \frac{A}{\alpha} \left(e^{-\alpha a} - e^{\alpha a} \right) \text{ for } |x| > a \\ &\quad \text{for } |x| < a \text{ more difficult!} \end{aligned}$$