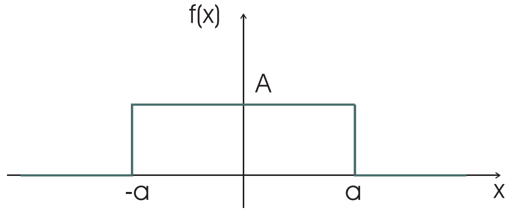
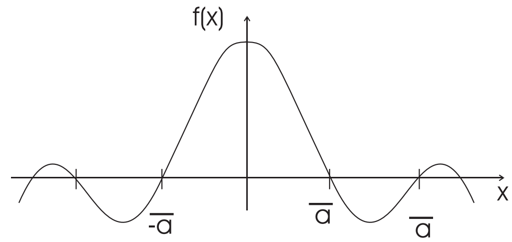
3.14.1 Examples for Fourier transformation

(i) $f(x) = \begin{cases} A & \text{for } |x| < a \\ 0 & \text{for} |x| > a \end{cases}$



$$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} A e^{-ipx} dx = \frac{A}{\sqrt{2\pi}} \frac{1}{-ip} \left[e^{-ipx}\right]_{-a}^{a}$$

$$F(p) = \frac{A}{\sqrt{2\pi}} \frac{2}{p} \underbrace{\frac{1}{2i}\left(e^{ipa} - e^{-ipa}\right)}_{\sin(pa)} = \frac{2aA}{\sqrt{2\pi}} \frac{\sin(pa)}{(pa)}$$



(ii)

$$f(x) \overset{\alpha \geq 0}{=} e^{-\alpha|x|}$$

$$\rightarrow F(p) = \frac{1}{\sqrt{2\pi}} \left(\int_0^{\infty} e^{-\alpha x} e^{-ipx} dx + \int_{-\infty}^{0} e^{\alpha x} e^{-ipx} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{1}{-(\alpha + ip)} e^{-(\alpha+ip)x} \right]_0^{\infty} + \left[\frac{1}{(\alpha - ip)} e^{(\alpha - ip)x} \right]_{-\infty}^{0} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\alpha - ip} + \frac{1}{\alpha + ip} \right) = \frac{1}{\sqrt{2\pi}} \frac{(\alpha + ip) + (\alpha - ip)}{\alpha^2 + p^2}$$

$$\Rightarrow F(p) = \sqrt{\frac{2}{\pi}} \alpha \frac{1}{\alpha^2 + p^2}$$

(iii)

$$f(x) = e^{-\frac{\alpha^2 x^2}{2}} \rightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{\alpha^2 x^2}{2}} e^{-ikx} dx$$

$$F(k) = \frac{1}{\alpha} e^{-\frac{k^2}{2\alpha^2}} \text{ FT(Gaussian) is a Gaussian}$$

(iv)

$$f(t) = \begin{cases} e^{-\alpha t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\alpha t} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{1}{-\alpha - i\omega} \left[e^{-\alpha t - i\omega t}\right]_0^{\infty}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + i\omega}$$