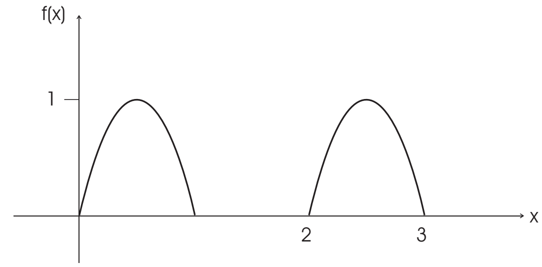
3.11.2 Example: Positive part of sin function

$$f(x) = \begin{cases} \sin x & 0 \le x < \pi \\ 0 & \pi \le x < 2\pi \end{cases} \quad \text{and periodic continuation}$$



$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^\pi \sin x dx = \frac{1}{2\pi}\left[-\cos x\right]_0^\pi = \frac{1}{\pi}$$

$$a_k = \frac{1}{\pi}\int_0^\pi \sin x \cos kx\, dx = \begin{cases} \frac{1}{\pi}\left[\frac{1}{2}\sin^2 x\right]_0^\pi & = 0, \text{ for } k=1 \\ \frac{1}{\pi}\left[-\frac{\cos(1+k)x}{2(1+k)} - \frac{\cos(1-k)x}{2(1-k)}\right]_0^\pi & = \begin{cases} 0 & k \text{ odd} \\ \frac{1}{\pi}\frac{2}{1-k^2} & k \text{ even} \end{cases}, \text{for } k>1 \end{cases}$$

$$b_k = \frac{1}{\pi}\int_0^\pi \sin x \sin kx\, dx = \begin{cases} \frac{1}{\pi}\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^\pi & = \frac{1}{2} \quad \text{for } k=1 \\ \frac{1}{\pi}\left[\frac{\sin(1-k)x}{2(1-k)} - \frac{\sin(1+k)x}{2(1+k)}\right]_0^\pi & = 0 \quad \text{for } k>1 \end{cases}$$

Thus: $$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)}\cos 2kx \quad \Rightarrow \quad \text{Due to } 1/(4k^2-1) \text{ fast converging series}$$

To solve the above integrals we used relations extracted directly from the addition theorems for sin- and cos-functions:

$$\sin(\alpha)\cos(\beta) = \tfrac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$$

$$\Rightarrow \sin(x)\cos(kx) = \tfrac{1}{2}\left(\sin((1-k)x)+\sin((1+k)x)\right)$$

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