3.6.2 Example: Taylor series of arctan function and π-calculation

$$\begin{aligned} \arctan 1 &= \frac{\pi}{4} \\ \arctan x &\stackrel{?}{=} \sum_{k=0}^{\infty} a_k x^k \qquad \text{Taylor-Series} \end{aligned}$$

We could calculate $f^n(0)$ but it may be simpler in the following very instructive way: We assume that $|x| < 1$

$$\begin{aligned} \arctan x &= \int_0^x \frac{1}{1+t^2} dt \ \leftarrow \text{note: } [\arctan x]' = \frac{1}{1+x^2} \\ &= \int_0^x \sum_{n=0}^{\infty} (-1)^n (t^2)^n dt \ \leftarrow \text{note:} \frac{1}{1-a} = \sum_{n=0}^{\infty} a^n,\ |a| < 1 \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

Thus:

$$\begin{aligned} \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \ \ ;\text{also convergent for } x = 1 \text{ (can be shown)} \\ x &= 1 \rightarrow\ \arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \\ &\rightarrow \ \text{very slow converging series: e.g. } 10^{10} \text{ terms for } \mathbf{10} \text{ precise digits of } \pi!! \end{aligned}$$

Much better: To exploit the properties of the arctan function

$$\begin{aligned} \frac{\pi}{4} &= 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \\ &= \frac{4}{5} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(\frac{1}{5}\right)^{2k} - \frac{1}{239} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(\frac{1}{239}\right)^{2k} \\ &\sim \ 10 \text{ terms for } > 10 \text{ precise digits of } \pi!! \quad \left(\frac{1}{5}\right)^{20} \sim 10^{-14} \end{aligned}$$

It remains to prove that

$$\arctan 1 = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

This we show in three steps by each time applying the addition theorem for the arctan function

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}.$$

$$\arctan 1 + \arctan \frac{1}{239} = \arctan \frac{1 + \frac{1}{239}}{1 - \frac{1}{239}} = \arctan \frac{120}{119}.$$

$$\arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \arctan \frac{5}{12}.$$

$$\arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan \frac{\frac{10}{12}}{1 - \frac{25}{144}} = \arctan \frac{10 * 20}{144 - 25} = \arctan \frac{120}{119}.$$

Combining these equation we easily get the desired result.