3.5.4 Example: Taylor expansion of the logarithm function

We will perform the Taylor expansion of the function

$$\begin{aligned}
f(x) &= \ln(1+x) \quad \text{around} \quad x_0 = 0 \\
f(0) &= \ln(1) = 0 \\
\frac{df}{dx} &= \frac{1}{1+x} \quad \Rightarrow \quad \frac{df}{dx}(0) = 1 \\
\frac{d^n f}{dx^n} &= \frac{(-1)(-2)\dots(-(n-1))}{(1+x)^n} \quad \Rightarrow \quad \frac{d^n f}{dx^n}(0) = (-1)^{n+1}(n-1)! \\
\Rightarrow f(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}
\end{aligned}$$

This result we use to prove one fundamental equation of the exponential function

$$\begin{aligned}
\ln\left(1+\frac{x}{l}\right)^l &= l \, \ln\left(1+\frac{x}{l}\right) = l \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{l^n \, n} \\
\Rightarrow \lim_{l\to\infty} \ln\left(1+\frac{x}{l}\right)^l &= x \\
\Rightarrow \lim_{l\to\infty} \left(1+\frac{x}{l}\right)^l &= \exp(x)
\end{aligned}$$