3.4.1 Examples: Convergence of infinite series

(i)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \rightarrow \quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n} = \frac{x}{n+1} \overset{n\to\infty}{\longrightarrow} 0$$

$$\rightarrow \quad \text{convergent for every } x$$

(ii)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \rightarrow \quad \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \rightarrow 1 \ ?$$

$$\sqrt[n]{a^n} = \frac{1}{\sqrt[n]{n^2}} = \frac{1}{\left(\sqrt[n]{n}\right)^2} \rightarrow 1 \ ?$$

$$\text{in fact} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ !$$

(iii)

$$\sum_{n=1}^{\infty} x^{n!} \qquad \frac{a_{n+1}}{a_n} = \frac{x^{(n+1)!}}{x^{n!}} = x^{(n+1)n!-n!} = x^{n\cdot n!} \rightarrow \begin{array}{l} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1 \\ \text{diverges if } |x| = 1 \end{array}$$