3.2.1 Prove of product rule and chain rule

We discuss the derivative of the product function $h(x) = f(x)g(x)$. So we find

$$\begin{aligned} h(x) - h(y) &= f(x)g(x) - f(y)g(y) \\ &= f(x)g(x) - f(x)g(y) - f(y)g(y) + f(x)g(y) \\ &= f(x)\left(g(x) - g(y)\right) + g(y)\left(f(x) - f(y)\right) \end{aligned}$$

and finally

$$\begin{aligned} \lim_{x\to y} \frac{h(x) - h(y)}{x - y} &= \lim_{x\to y} f(x)\frac{g(x) - g(y)}{x - y} + \lim_{x\to y} g(y)\frac{f(x) - f(y)}{x - y} \\ &= f(x)\frac{dg}{dx}(x) + g(x)\frac{df}{dx}(x) \end{aligned}$$

Second we discuss the derivative of $h(t) = g(f(t))$. According to the definition of the derivative two functions $u(t)$ and $v(t)$ exist with $u(0) = v(0) = 0$ and

$$\begin{aligned} f(t) - f(x) &= (t - x)\left[\frac{df}{dx}(x) + u(t)\right] \\ g(s) - g(y) &= (s - y)\left[\frac{dg}{dy}(y) + v(s)\right] \end{aligned}$$

so

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)]\left[\frac{dg}{dy}(y) + v(s)\right] \\ &= (t - x)\left[\frac{df}{dx}(x) + u(t)\right]\left[\frac{dg}{dy}(y) + v(s)\right] \end{aligned}$$

and finally

$$\frac{dh}{dx} = \lim_{t\to x} \frac{h(t) - h(x)}{t - x} = \frac{df}{dx}(x)\frac{dg}{dx}(f(x))$$