2.15.1 Mathematical description of a plane wave

The mathematical description of a plane wave combines the properties of the scalar product, the vector product, and the complex exponential function; so in can serve as a very good example to illustrate the geometrical properties of this three concepts.
The most illustrative way to define the set of vectors pointing to a plane wave is given by

$$\vec{r} = \vec{r}_0 + \alpha\vec{a} + \beta\vec{b};$$

here $\vec{r}_0$ is a vector from the origin to a point of the plane, $\vec{a}$ and $\vec{b}$ are two non parallel vectors lying in the plane and α and β are two real numbers.
A normal vector of the plane can be calculated by

$$\vec{N} = \vec{a} \times \vec{b}.$$

This normal vector can be used to define the set of vectors pointing to the plane by

$$\left\langle \vec{N} \middle| \, \vec{r} \right\rangle = \left\langle \vec{N} \middle| \, \vec{r}_0 \right\rangle .$$

i.e. all vectors pointing to the plane have the same projection onto the normal vector of the plane.
A moving plane thus is represented by

$$\left\langle \vec{N} \middle| \, \vec{r} \right\rangle = \left\langle \vec{N} \middle| \, \vec{r}_0 \right\rangle + v\, t \quad ;$$

here v is the speed of the plane in the direction of the normal vector.
A plane wave typically is written as

$$\exp i \, \left(\left\langle \vec{k} \middle| \, \vec{r} \right\rangle - \omega\, t \right) = \exp i \, \left(\vec{k}\, \vec{r} - \omega\, t \right) \quad ;$$

here $\vec{k}$ is again the normal vector to the plane (the length is $k = \left|\vec{k}\right| = \frac{2\pi}{\lambda}$, λ: wave lenght), $\omega = \frac{2\pi}{T}$ (T: period), and the velocity of the plane wave is $c = \frac{\omega}{k}$.