2.5 Some formal definitions in linear Algebra

Although we will not use the following definitions as a starting point for discussing vectors, later on there are some aspects which will need a more general point of view. For sake of completeness and correctness we therefor will already introduce here some formal definitions.

Definition 11 *A commutative group* $(G, +)$ *is defined by*

Closure:	*For all* a *and* b *in* G*,* $a + b$ *belongs to* G.
Associativity:	*For all* a*,* b*, and* c *in* G*,* $(a + b) + c = a + (b + c)$.
Neutral element:	*There is an element* e *in* G *such that for all* a *in* G*,* $e + a = a + e = a$.
Inverse element:	*For each* a *in* G*, there is an element* b *in* G *such that* $a + b = b + a = e$*, where* e *is the neutral element from the previous axiom.*
Commutative	*For all* a *and* b *in* G*,* $a + b = b + a$.

Definition 12 *Let* $(G, +)$ *be a commutative group and* $a, b \in G$*. Let* $(K, +, *)$ *be a field and* $\alpha, \beta \in K$*.* V *is a vector space, if*

Distributive 1:	$\alpha(a + b) = \alpha a + \alpha b$.
Distributive 2:	$(\alpha + \beta)a = \alpha a + \beta a$.
Multiplications are compatible:	$(\alpha\ \beta)\ a = \alpha\ (\beta\ a)$.
Scalar multiplication has an identity element:	$1\ a = a$.

Definition 13 *Let* V *be a vector space. A transformation* $A: V \rightarrow V$ *is a linear function, if for all* $f_1,\ f_2 \in V$ *and all* $\lambda \in \mathbb{R}\ (\mathbb{C})$

Linearity 1:	$A\ (f_1\ +\ f_2) = A\ f_1\ +\ A\ f_2$.
Linearity 2:	$A\ (\lambda\ f_1) = \lambda\ A(f_1)$.

Definition 14 *Let* V *be a vector space over a field* K*. A transformation* $\langle\ ,\ >: V \times V \rightarrow K$ *is a scalar product, if*

Nearly Commutative:	$\langle x, y\rangle = \langle y, x\rangle^*$.
Linearity 1:	$\langle x_1\ +\ x_2, y\rangle = \langle x_1, y\rangle\ +\ \langle x_2, y\rangle$.
Linearity 2:	$\langle x, \alpha\ y\rangle = \alpha\ \langle x, y\rangle$.
Positive:	$\langle x, x\rangle \in \mathbb{R}^+$ *for* $x \neq 0$.

This last definition holds for vectors with complex components. For real numbers the $*$ in the commutative law can be omitted.