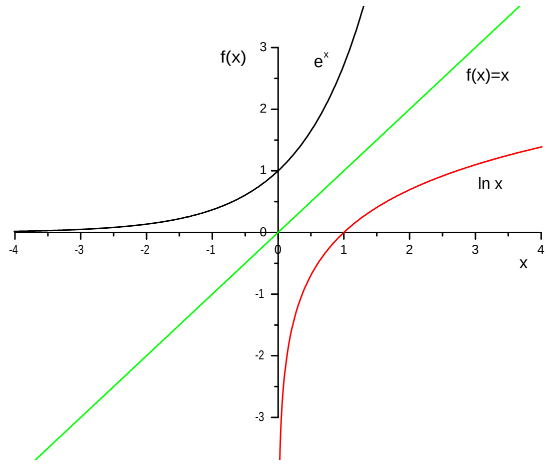
2.4 Other complex functions

Logarithm:
real values: $f(x) = e^x$



$$w = e^x \to x = \ln w \qquad \Rightarrow f^{-1}(x) = \ln x \text{ is the inverse function with respect to } e^x$$

$$\ln 1 = 0 \qquad \ln e = 1 \qquad ln0 = -\infty$$

now complex:

$$\begin{aligned} w = e^z &= \left(e^a \cdot e^{ib}\right) \leftarrow 2\pi \text{ periodic in b} \\ &\Leftrightarrow \quad z = \log w = \ln|w| + i\varphi + 2\pi k i, \; k \in \mathbb{Z} \end{aligned}$$

test:

$$e^z = e^{\ln|w| + i\varphi + 2\pi k i} = e^{\ln|w|} \cdot e^{i(\varphi + 2\pi k)} = |w|e^{i\varphi} = w \Rightarrow \text{o.k.}$$

Definition 9

$$f(z) = \log(z) = \ln|z| + i\varphi + 2\pi k i, \; k \in \mathbb{Z} \; z = |z|e^{i\varphi}$$

is the complex logarithm of z. For $k = 0$ we get the main value.

Example:

$$\begin{aligned} \ln(-1) &= \underbrace{\ln 1}_{0} + i \underbrace{\varphi}_{\pi} + 2\pi k i = i\pi + 2\pi k i = i\pi(2k+1) \\ &\to \text{logarithm of negative numbers now defined} \\ &\quad \ln 0 \text{ still } -\infty \end{aligned}$$

Exponential function with arbitrary base:
real numbers: $a^b = e^{b \ln a} \qquad a > 0$
generalization for complex numbers: $b \log a \qquad b \in \mathbb{C}, a \in \mathbb{C}\backslash\{0\} \Rightarrow a^b = e^{b \log a} = e^{b(\ln|a| + i\varphi_a + 2\pi i k)}$

Definition 10 $0^0 = 1$

special function:

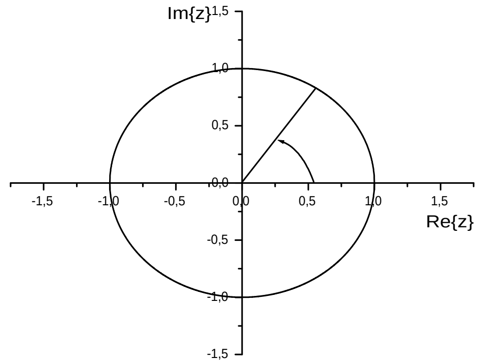
$$\begin{aligned} f(z) &= a^z, \quad a > 0 \quad (a = e\text{-function}) \\ f(z) &= e^{z \log a} = e^{\ln a \ \mathrm{Re}(z)} e^{i \ln a \ \mathrm{Im}(z)} \quad \text{o.k.} \end{aligned}$$

important (for root finding):

$$\begin{aligned} f(z) &= z^b, \quad b \in \mathbb{R} \qquad z = |z|e^{i\varphi} \\ \to f(z) &= e^{b \log z} = e^{b(\ln|z| + i\varphi + 2\pi i k)} \\ &= e^{b(\ln|z| + i\varphi)} e^{2\pi i k b}; \quad k \in \mathbb{Z} \\ \text{different cases: } b \in \mathbb{Z} &\to e^{2\pi i k b} = 1 \to \text{ o.k. ''normal'' power of z} \\ b \text{ irrational} &\to e^{2\pi i k b} \text{ infinite number of values} \\ b \text{ rational} &\to e^{2\pi i k b} \,\hat{=}\, \text{finite number of values} \end{aligned}$$

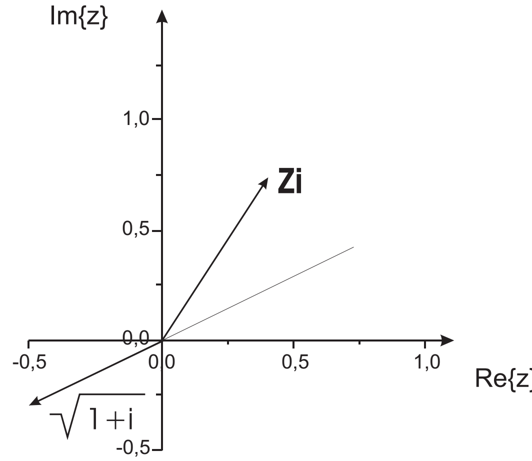
Example:

$$b = \frac{1}{n} \; : \; e^{2\pi i \frac{k}{n}} \; k = 0, \ldots, n-1$$
$$\Rightarrow z^{\frac{1}{n}} = |z|^{\frac{1}{n}} e^{i\frac{\varphi}{n}} e^{2\pi i \frac{k}{n}} \quad k = 0, \ldots, n-1$$



in particular:

$$n = 2 \; : \; \sqrt{1+i} = \sqrt{2}^{\frac{1}{2}} e^{i\frac{\pi}{4}\frac{1}{2}} e^{2\pi i \frac{k}{n}} \; , k = 0, 1$$
$$\text{for } k = 0 \quad : \quad \sqrt{2}^{\frac{1}{2}} e^{i\frac{\pi}{8}} e^{0} = \sqrt{2}^{\frac{1}{2}} e^{i\frac{\pi}{8}}$$
$$\text{for } k = 1 \quad : \quad \sqrt{2}^{\frac{1}{2}} e^{i\frac{\pi}{8}} e^{\pi i} = \sqrt{2}^{\frac{1}{2}} e^{\pi i (1+\frac{1}{8})} = \sqrt{2}^{\frac{1}{2}} e^{\frac{9}{8} i\pi}$$



$\Rightarrow$ Roots are difficult functions, in particular not single values.
The most simple case we know already from square roots of pure positive real numbers a. The solution is

$$\pm\sqrt{a} = e^{\frac{2\pi i k}{2}} \sqrt{a}$$

which for $k \in \mathbb{Z}$ has the two independent solutions for $k = 0$ and $k = 1$.

Fundamental theorem of algebra

Each polynomial equation of degree $n \in \mathbb{N}$

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

with $a_0, \ldots, a_n \in \mathbb{C}$ has at least one solution $z \in \mathbb{C}$.
$\Rightarrow$ Say this is z_1, then:

$$(z - z_1) \cdot \left(b_{n-1} z^{n-1} + \ldots + b_0\right) = 0$$

$\Rightarrow$ each polynomial has exactly n solutions where so-called "multiple zeros", i.e. function $(z - z_0)^k$ count k-times.

$n = 1$	$a_1 z + a_0 = 0$	$\Rightarrow$	$z = -\frac{a_0}{a_1}$
$n = 2$	$a_2 z^2 + a_1 z + a_0 = 0$	$\Rightarrow$	quadratic equation
$n = 3$	$a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$	$\Rightarrow$	solution via Cardano formula
$n = 4$	$a_4 z^4 + \ldots + a_0 = 0$	$\Rightarrow$	solution via formula possible
$n > 4$		$\Rightarrow$	no formula exist for a general treatment!