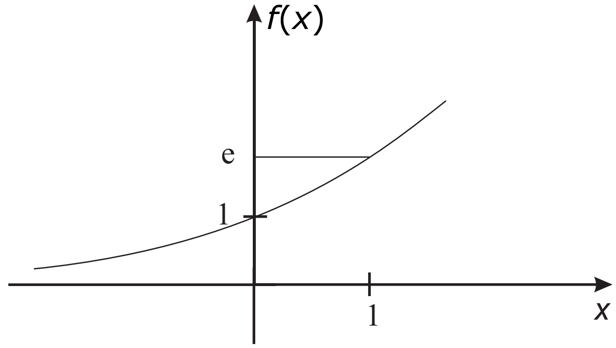
2.3 Complex e-function

Function: $f(x) = e^x,\ x \in \mathbb{R}, \quad e = 2.7181\ldots$



We define the exponential function as (the only non trivial function) which is it's own derivative:
Derivative of the exponential function:

$$\frac{de^x}{dx} = e^x$$

Just using the definition of the factorial function we find the Taylor series expansion of the exponential function

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \text{Euler's number: } e$$

First we will prove some very important properties of the exponential function.
Fundamental addition formula of the exponential function:

Applying the definition of the e-function as a series we find

$$\begin{aligned} e^x e^y &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\ &= e^{x+y} \end{aligned}$$

Euler's formula:
taking into account $i^{4k} = 1, \quad i^{4k+1} = i, \quad i^{4k+2} = -1, \quad i^{4k+3} = -i$, and using the definitions by the Taylor series we find

$$\begin{aligned} e^{i\varphi} &= \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!} = 1 + i\varphi + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \frac{(i\varphi)^5}{5!} + \ldots \\ &= 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots + i\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \ldots\right) \\ &:= \cos\varphi + i\sin\varphi \end{aligned}$$

This is a very important relation. It can be understood as the definition of the sin and cos function and allows to replace $\cos\varphi$ and $\sin\varphi$ by the (complex) e-function (and vice versa) $\Rightarrow$ Simplification!!! (e.g. Waves $\hat{=}$ complex e-function). In addition the symmetries of the sin and cos functions get already obvious.
$\sin x$ and $\cos x$ vs. exp:

For real numbers $\cos x$ and $\sin x$ are just the symmetric resp. antisymmetric representation of the $\exp x$ function with the following properties

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \qquad x \in \mathbb{R} \\ e^{-ix} &= \cos x - i \sin x \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \cos x &= \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) \\ \sin x &= \tfrac{1}{2i}\left(e^{ix} - e^{-ix}\right) \end{aligned}$$

Additionally we directly get

$$\cos^2 x + \sin^2 x = (\cos x + i \sin x)(\cos x - i \sin x) = e^{ix} e^{-ix} = 1$$

<u>Addition theorems for sin and cos functions:</u>

Combining the exp-addition formula with Euler's formula we find

$$\begin{aligned} (\cos y \cos z - \sin y \sin z) + i (\cos y \sin z + \sin y \cos z) &= \\ (\cos y + i \sin y)(\cos z + i \sin z) &= e^{iy} e^{iz} \\ &= e^{i(y+z)} \\ &= \cos(y+z) + i \sin(y+z) \end{aligned}$$

From real and imaginary part we finally get (representing the even and odd part of the complex exponential function)

$$\begin{aligned} \cos y \cos z - \sin y \sin z &= \cos(y+z) \\ \cos y \sin z + \sin y \cos z &= \sin(y+z) \end{aligned}$$

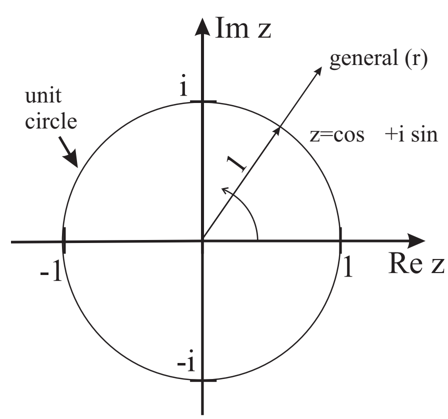
Combining both equations we easily get

$$\tan(y+z) = \frac{\tan(y) + \tan(z)}{1 - \tan(y)\,\tan(z)}$$

<u>Back to complex numbers:</u>

→ <u>In general:</u>

$$\begin{aligned} z &= r(\cos\varphi + i \sin\varphi) \\ \mathrm{Re}\{z\} &= r \cos\varphi \\ \mathrm{Im}\{z\} &= r \sin\varphi \\ r &= |z| \end{aligned}$$



<u>Multiplication of complex numbers:</u>

$$\begin{aligned} z_1 \cdot z_2 = r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} &= (r_1 r_2) e^{i(\varphi_1 + \varphi_2)} \\ &= (r_1 r_2)(\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)) \\ z \cdot \bar{z} = r e^{i\varphi} \cdot r e^{-i\varphi} &= r^2 \end{aligned}$$

Definition 6

$$f(z) = e^z \quad z \in \mathbb{C} \textit{ is the complex e-function with}$$

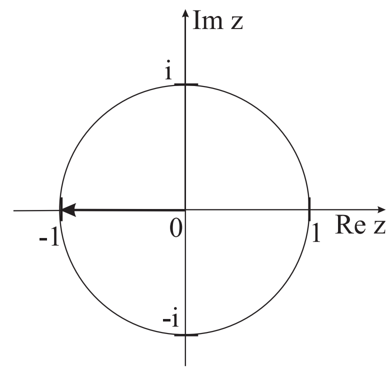
$$z = a + bi \;\Rightarrow\; e^{a+bi} = e^a e^{bi} = e^a(\cos b + i \sin b) \;\Rightarrow \textit{complex e-function is periodical in } 2\pi$$

$$\begin{aligned} Re(e^z) &= e^a \cos b; \\ Im(e^z) &= e^a \sin b; \\ b = 0 &\to e^z = e^a \;\; o.k.; \\ a = 0 &\to e^z = e^{ib} = \cos b + i \sin b \;\; o.k.; \end{aligned}$$

Example:

Has the equation $e^z = -1$ any solution? (see also 3.5)

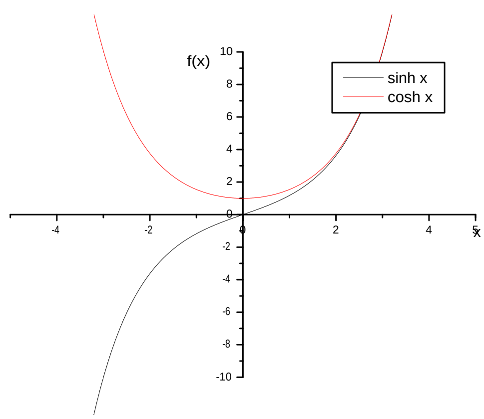
$$\begin{aligned} \text{z-real} &\to \text{no} \\ \text{z-complex} \Rightarrow e^a \cos b &= -1 \\ e^a \sin b &= 0 \Rightarrow b = n\pi \\ &\Rightarrow \underbrace{e^a}_{a=0} \underbrace{\cos n\pi}_{n=2k+1} = -1 \\ &\Rightarrow z = (2k+1)\pi i \\ &\to e^{\pi i} + 1 = 0 \;\; \text{beautiful expression} \end{aligned}$$



Similar to the definition of the cos and sin function we have

Definition 7 *hyperbolic functions*

$$\begin{aligned} \cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \in \mathbb{R} \\ \sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right) \\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$



like tan $x \to$ *Definitions also valid for complex arguments*

$$\begin{aligned} \sinh z &= \frac{1}{2}\left(e^z - e^{-z}\right), \\ \cosh z &= \frac{1}{2}\left(e^z + e^{-z}\right), \end{aligned}$$

$$\begin{aligned} \sinh(ix) &= \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i \sin x, \\ \cosh(ix) &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos x \end{aligned}$$

Theorem for cosh *and* sinh*:*

$$\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x} = 1$$

$\to \sin z, \cos z$ for complex arguments are also defined in a logical way:

Definition 8

$$
\begin{aligned}
\sin z &= \tfrac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
\cos z &= \tfrac{1}{2}\left(e^{iz} + e^{-iz}\right)
\end{aligned}
\qquad z \in \mathbb{C}
$$

e.g.:

$$
\begin{aligned}
\sin z = 2 &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
\Rightarrow 4i &= \underbrace{e^{iz}}_{w} - e^{-iz} \\
\Leftrightarrow w^2 - 4iw - 1 &= 0 \\
\Rightarrow w_{1/2} &= 2i \pm \sqrt{5}i \\
w = e^{iz} &= (2 \pm \sqrt{5})i \\
\to \mathrm{Re}(z) &= 0 \to \cos b = 0 \to b = \left(n + \frac{1}{2}\right)\pi \\
\to \sin b &= \pm 1 \\
\pm e^{a} &= \left(2 \pm \sqrt{5}\right) \\
a &= \ln|2 \pm \sqrt{5}| \\
\Rightarrow z &= \ln|2 \pm \sqrt{5}| + i\left(n + \frac{1}{2}\right)\pi
\end{aligned}
$$

The above relation between sin, cos, sinh, and cosh allow e.g. to easily apply the addition theorems to calculate

$$
\begin{aligned}
\cosh(a + ib) &= \cosh(a)\cosh(ib) + \sinh(a)\sinh(ib) \\
&= \cosh(a)\cos(b) + i\sinh(a)\sin(b)
\end{aligned}
$$