4.11 Integrals using grad, div, and curl

We already defined the nabla operator in connection with the gradient in section 39

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_N} \end{pmatrix} \qquad \vec{x} \in \mathbb{R}^N$$

Applying this operator to a function with one component $f(\vec{x})$ we get the gradient

$$\text{grad} f = \vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{pmatrix}.$$

Note that the gradient is applied to a scalar and the result is a vector.
One essential aspect of the gradient is the solution of path integrals

$$\int\limits_{\vec{x}_b}^{\vec{x}_e} \vec{\nabla} f \; d\vec{x} = f(\vec{x}_e) - f(\vec{x}_b).$$

Calculating the scalar product of the nabla operator and a function with several components $\vec{f}(\vec{x})$ we get the divergency

$$\text{div}\vec{f} = \vec{\nabla}\vec{f} = \sum_i \frac{\partial f_i}{\partial x_i}.$$

Note that the divergency is applied to a vector and the result is a scalar.
One essential aspect of the divergency is the solution of volume integrals

$$\iiint\limits_V \vec{\nabla}\vec{f} \; dx \; dy \; dz = \oiint\limits_{\partial V} \vec{f} \; d\vec{A}.$$

Here ∂V denotes the closed surface of the volume V the integration is calculated over.
Calculating the vector product of the nabla operator and a function with several components $\vec{f}(\vec{x})$ we get the curl

$$\text{curl}\vec{f} = \text{rot}\vec{f} = \vec{\nabla} \times \vec{f}.$$

Note that the curl is applied to a vector and the result is a vector.
One essential aspect of the curl is the solution of area integrals (Stokes integral equation)

$$\iint\limits_A \vec{\nabla} \times \vec{f} d\vec{A} = \oint\limits_{\partial A} \vec{f} \; d\vec{x}.$$

Here ∂A denotes the closed path around the area A the integration is calculated over.

Examples using the Maxwell equations:

As an example for a vector function we already discussed in section 4.2 the electric field of a point source with positive charge q, i.e. the charge density $\rho(\vec{r}) = q\delta(\vec{r})$. The electrical field strength is calculated from the 1. Maxwell equation

$$\frac{\rho}{\epsilon_0} = \vec{\nabla}\vec{E}(\vec{r}).$$

Integrating the Maxwell equation over a sphere with radius r centered around the point charge we find

$$\frac{q}{\epsilon_0} = \iiint\limits_{\text{sphere}} \frac{q\delta(\vec{r})}{\epsilon_0} \; dx \; dy \; dz = \iiint\limits_{\text{sphere}} \vec{\nabla}\vec{E}(\vec{r}) \; dx \; dy \; dz = \oiint\limits_{\text{surface sphere}} \vec{E}(\vec{r}) \; d\vec{A}.$$

Obviously the electrical field strength $\vec{E}(\vec{r}) = E(r)\frac{\vec{r}}{r}$ has a radial symmetry; thus the right hand integral can be simplified

$$\oiint\limits_{\text{surface sphere}} \vec{E}(\vec{r}) \; d\vec{A} = \oiint\limits_{\text{surface sphere}} E(r) \; dA,$$

i.e. the scalar product reduces just to the product of the length of vectors, finally leading to

$$\frac{q}{\epsilon_0} = E(r) 4\pi r^2,$$

where $4\pi r^2$ is the surface area of a sphere with radius r, leading to

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\frac{\vec{r}}{r}.$$

Since $\vec{E} = -\vec{\nabla} U$ and taking the potential $U(\infty) = 0$ we find the potential of a point charge as

$$U(\vec{r}) = \int_r^\infty (-\vec{\nabla} U)\, d\vec{r} = \int_r^\infty \vec{E}\, d\vec{r} = \int_r^\infty E\, dr = \frac{q}{4\pi\epsilon_0 r}.$$

For this final result the path along the $\vec{r}$ direction has been chosen.

The 2. Maxwell equation reads

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0\, \vec{j}(\vec{r}).$$

Having a straight wire with a current density $\vec{j}$ we find according to the Stokes integral equation

$$\mu_0\, I = \iint_A \mu_0 \vec{j}(\vec{r}) d\vec{A} = \iint_A \vec{\nabla} \times \vec{B} d\vec{A} = \oint_{\partial A} \vec{B}\, d\vec{x}.$$

Choosing for the path in the right hand side integral a circle perpendicular to the wire centered around the center of the wire, which due to symmetry implies $\vec{B}\, d\vec{x} = const.$ along the path, we finally get

$$\mu_0\, I = |\vec{B}| 2\pi r.$$