2.2 Addition and Multiplication of complex numbers

Definition 4

$$\begin{array}{l} z_1 = a_1 + b_1 i \\ z_2 = a_2 + b_2 i \end{array} \Rightarrow \begin{array}{lcl} \oplus z_1 + z_2 &=& (a_1 + a_2) + (b_1 + b_2)i \\ \ominus z_1 - z_2 &=& z_1 + (-z_2) = (a_1 - a_2) + (b_1 - b_2)i \\ \odot z_1 \cdot z_2 &=& (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + b_1 a_2 i + a_1 b_2 i + b_1 b_2 i^2 \\ &=& (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i \end{array}$$

e.g.:

$$(3 + 2i)(1 - i) = 3 - 3i + 2i + 2 = 5 - i$$

$$(a + bi) + \overbrace{0}^{0+0i} = a + bi \tag{2.9}$$

$$(a + bi) \cdot \underbrace{1}_{1+0i} = a + bi \tag{2.10}$$

division: meaning of $\frac{1}{1+i}$ =?

$$\Rightarrow \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i \quad \text{o.k.}$$

<u>In general:</u>

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

$$z = a + bi \to z \cdot z^{-1} = 1 \Leftrightarrow z^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

Definition 5 *$\bar{z} = a - bi$ is the conjugate complex number to $z = a + bi$, other notation $\bar{z} = z^*$*

Summary:

$$\begin{array}{lcl} z &=& a + bi \\ \mathrm{Re}(z) &=& a \\ \mathrm{Im}(z) &=& b \\ \mathrm{Re}(\bar{z}) &=& a \\ \mathrm{Im}(\bar{z}) &=& -b \end{array} \quad \Rightarrow \quad \begin{array}{l} \mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}) \\ \\ \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z}) \end{array}$$

<u>Rules:</u> $\bar{\bar{z}} = z$, $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$, $\overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$
"Normal" rules:

- <u>commutative</u> law for addition and multiplication

$$z_1 + z_2 = z_2 + z_1 \tag{2.11}$$

$$z_1 z_2 = z_2 z_1 \tag{2.12}$$

- <u>associative</u> law $\oplus\odot$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \tag{2.13}$$

$$z_1(z_2 z_3) = (z_1 z_2) z_3 \tag{2.14}$$

- <u>distributive</u> law $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
- <u>neutral</u> element

$$\begin{array}{ll} \oplus & 0 \\ \odot & 1 \end{array}$$

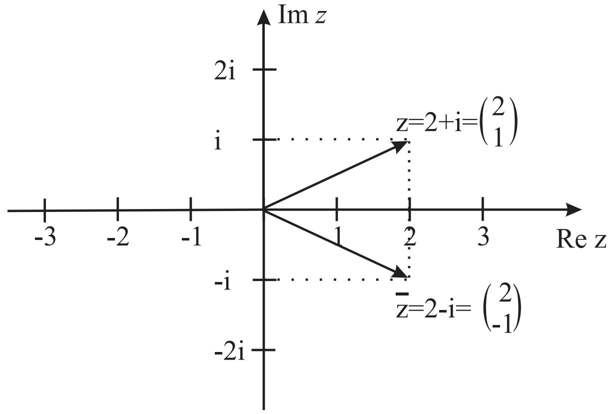
- unique solutions $z_1 = v + z_2 \quad \Rightarrow \quad (v = z_1 - z_2)$
- unique solutions $z_1 = w z_2 \quad \Rightarrow \quad (w = z_1 / z_2)$
- <u>no</u> ordering

order properties: $2 < 3,\ 0 < 1$ o.k.
but $i \gtrless 0$?

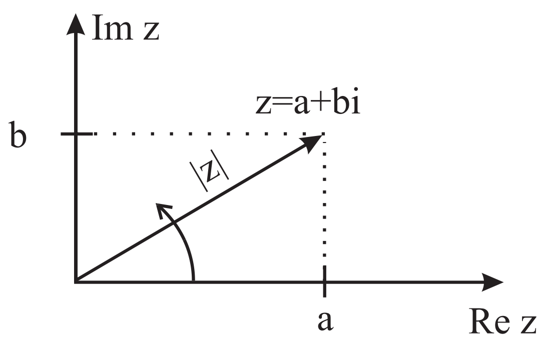
$$i > 0 \to i \cdot i > 0 \cdot i = 0 \to -1 > 0 \quad \wr \tag{2.15}$$

$$i < 0 \to i \cdot i^3 < 0 \cdot i^3 = 0 \to 1 < 0 \quad \wr \tag{2.16}$$

$\Rightarrow$ nonsense, complex numbers do not have order at all! (later we will discuss the modulus of z)
Gauß Plane of Numbers



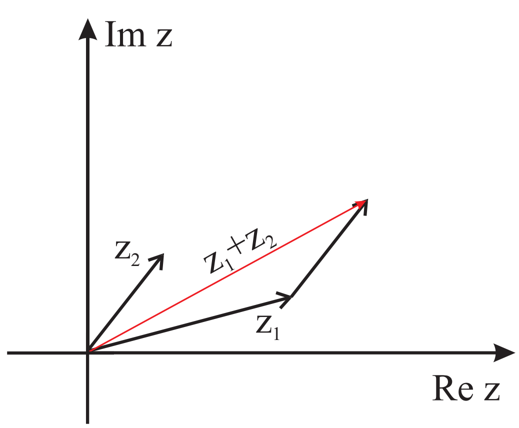
$\Rightarrow$ complex numbers are 2D vectors with certain properties



Modulus $r = |z|$ and phase φ

$$\begin{aligned} z &= (r, \varphi), \\ r &= \sqrt{a^2 + b^2},\ r = |z| = \sqrt{z\bar{z}}, \\ \varphi &= \arctan\left(\frac{b}{a}\right) \to \varphi \text{ in radians!!} \end{aligned}$$

$z_1 \lessgtr z_2$ makes no sense, but $|z_1| \lessgtr |z_2|$ is fine!
e.g.: $z_1 = 2 + i = (\sqrt{5}, \arctan(\frac{1}{2}))$ or $z_2 = 1 + i = (\sqrt{2}, \frac{\pi}{4})$
Geometrical interpretation of + and $\cdot$ with complex numbers:



$$\begin{aligned} z_1 &= (r_1, \varphi_1) & z_2 &= (r_2, \varphi_2) \\ &= a_1 + b_1 i & &= a_2 + b_2 i \\ z_1 + z_2 &= \text{"Adding 2D vectors"} \\ & \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} \\ z_1 \cdot z_2 &= (r_1 r_2, \varphi_1 + \varphi_2) \end{aligned}$$

$\hat{=}$ Rotation and stretching of z_1 by z_2
$\Rightarrow$ see later

e.g.: $(1 + i)(\frac{1}{2} + \frac{1}{2}i) = (\sqrt{2}\frac{1}{\sqrt{2}}, \frac{\pi}{4} + \frac{\pi}{4}) = (1, \frac{\pi}{2}) = i$