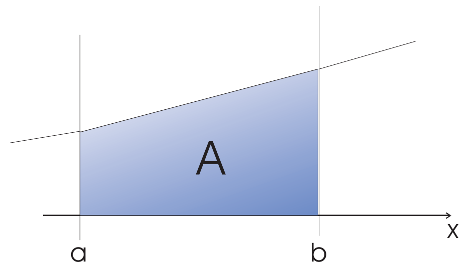
4.9 Simple N-dimensional integrals

$$\left.\begin{array}{c}\int\limits_a^b f(x)dx = A = F(b) - F(a) \\ \text{with } \frac{dF}{dx} = f(x)\end{array}\right\} \text{1D case}$$

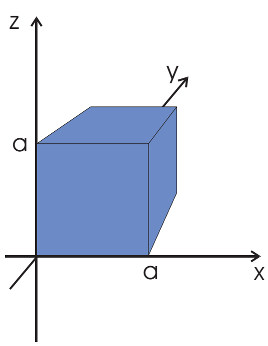


We will only discuss a 3D example: $f : \mathbb{R}^3 \to \mathbb{R} \;\; f(x, y, z)$
Integrals over volumes:

$$\iiint\limits_V f(x,y,z)dx\;dy\;dz = \int\limits_{x_1}^{x_2}\left(\int\limits_{y_1(x)}^{y_2(x)}\left(\int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right)dy\right)dx$$

It is often difficult to determine the border, the integrals may be simple. Simplest case: Cartesian limits
Example: $f(x, y, z) = yxz$ integral over a cube with edge a

$$\begin{aligned}\iiint\limits_V f(x,y,z)dx\;dy\;dz &= \int\limits_0^a\left(\int\limits_0^a\left(\int\limits_0^a xyz\;dz\right)dy\right)dx \\ &= \int\limits_0^a\int\limits_0^a\left(\left[xy\frac{1}{2}z^2\right]_0^a dy\right) = \int\limits_0^a\left[x\frac{1}{4}y^2a^2\right]_0^a dx = \left[\frac{1}{8}x^2a^4\right]_0^a = \frac{1}{2^3}a^6\end{aligned}$$

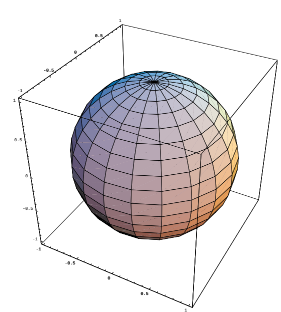


Volume: $f(x, y, z) = 1 \to \iiint\limits_{\bigcirc} 1dx\;dy\;dz \,\hat{=}$ Volume of the integration area

Example: Volume of a sphere with radius R, i.e. $R^2 = x^2 + y^2 + z^2$

$$\iiint\limits_{\bigcirc} 1\;dx\;dy\;dz \;= ?$$

$x \in [-R, R]$
$y \in \left[-\sqrt{R^2 - x^2}, +\sqrt{R^2 - x^2}\right]$
$z \in \left[-\sqrt{R^2 - x^2 - y^2}, +\sqrt{R^2 - x^2 - y^2}\right]$



Just writing down the limits for the integral is tedious work in Cartesian coordinates, but very simple in spherical coordinates:

$$V = \iiint\limits_{\bigcirc} 1\;dx\;dy\;dz = \int\limits_0^{\pi}\int\limits_0^{2\pi}\int\limits_0^{R} r^2\sin(\vartheta)\;dr\;d\varphi\;d\vartheta$$

Here we used the Jacobi determinant as calculated in section 4.5.1 to get the correct integration scaling for this coordinate transformation. Finally we get

$$V = 2\pi \left(-\cos(\vartheta)|_0^\pi \right) \left(\frac{r^3}{3} \bigg|_0^R \right) = \frac{4}{3}\pi R^3$$