4.8 Criteria for finding extreme values in N-dimensions

Definition 42 *If $\tilde{A}$ is a symmetric $N \times N$ matrix, $\vec{x} \in \mathbb{R}^N$ than:*
$\tilde{A}$ is positive definite if $\vec{x} \cdot \tilde{A}\vec{x} > 0$ for all $\vec{x}$
$\tilde{A}$ is positive semi-definite if $\vec{x} \cdot \tilde{A}\vec{x} \geq 0$ for all $\vec{x}$
$\tilde{A}$ is negative definite if $\vec{x} \cdot \tilde{A}\vec{x} < 0$ for all $\vec{x}$
$\tilde{A}$ is negative semi-definite if $\vec{x} \cdot \tilde{A}\vec{x} \leq 0$ for all $\vec{x}$
$\tilde{A}$ is called indefinite if $\vec{x}, \vec{y}$ exist with $\vec{x} \cdot \tilde{A}\vec{x} > 0$ and $\vec{y} \cdot \tilde{A}\vec{y} < 0$

Criterion for extreme values in N-dimensions:
$f : \mathbb{R}^N \to \mathbb{R}$ $\vec{x}_0 \in \mathbb{R}^N$ extreme value (local) than $\vec{\nabla} f(\vec{x}_0) = 0$
if:
$\tilde{H}(\vec{x}_0)$ positive definite$\rightarrow$ minimum
$\tilde{H}(\vec{x}_0)$ negative definite$\rightarrow$ maximum
$\tilde{H}(\vec{x}_0)$ indefinite$\rightarrow$ no extremum (saddle point)
$\tilde{H}(\vec{x}_0)$ positive semi-definite or neg. semi-definite then no decision is possible
Note: 1D is special case.

Hurwitz criterion: $\tilde{A}$ $N \times N$ matrix, symmetric

$$\tilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \quad \text{is positive definite if and only if} \quad \det \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0$$

$$\text{for all } k = 1, \ldots, N \ .$$

$\tilde{A}$ is negative definite if $-\tilde{A}$ is positive definite according to the Hurwitz criterion.
Otherwise $\tilde{A}$ is indefinite, or semi-case.

Other criteria:
$\tilde{A}$ symmetric, positive definite $\Rightarrow$ all Eigenvalues positive
$\tilde{A}$ symmetric, negative definite $\Rightarrow$ all Eigenvalues negative
$\tilde{A}$ symmetric semi cases$\Rightarrow$ $\lambda = 0$ possible
otherwise indefinite