4.6 Curvilinear coordinates

Spherical and cylindrical coordinates are examples of curvilinear coordinates u_1, u_2, and u_3 for which at each point holds

$$d\vec{r} = \sum_{i=1}^{3} a_i(u_1, u_2, u_3)\, \vec{e}_i(u_1, u_2, u_3)\, du_i \quad .$$

with $\vec{e}_i \vec{e}_k = \delta_{i\,k}$, i.e. curvilinear coordinates form locally an orthogonal base. The base vectors have a length $a_i(u_1, u_2, u_3)$ which depends on u_k.
Consequently the Jacobi matrix and determinant and their inverses are

$$J = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ a_1\vec{e_1} & a_2\vec{e_2} & a_3\vec{e_3} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \quad \det(J) = a_1\, a_2\, a_3 \quad ; \quad J^{-1} = \begin{pmatrix} \leftarrow & \frac{\vec{e_1}}{a_1} & \rightarrow \\ \leftarrow & \frac{\vec{e_2}}{a_2} & \rightarrow \\ \leftarrow & \frac{\vec{e_3}}{a_3} & \rightarrow \end{pmatrix} \quad \det(J^{-1}) = \frac{1}{a_1\, a_2\, a_3}$$

According to the chain rule the gradient in curvilinear coordinates can be written as

$$\text{grad} f = \vec{\nabla} f = \sum_{i=1}^{3} \frac{\partial f}{a_i\, \partial u_i} \vec{e}_i \quad .$$

It is hard work to find a general expressing for the Laplace operator in curvilinear coordinates. Still we will outline the prove since it summarizes nearly everything we learned about linear algebra and analysis.

$$\begin{aligned} \text{div grad} f = \Delta f &= \sum_{i,k=1}^{3} \frac{\langle e_k |}{a_k} \frac{\partial}{\partial u_k} | e_i \rangle \left(\frac{\partial f}{a_i \partial u_i} \right) \\ &= \sum_{i,k} \frac{\langle e_k | \, | e_i \rangle}{a_k} \frac{\partial}{\partial u_k} \left(\frac{\partial f}{a_i \partial u_i} \right) + \sum_{i,k} \frac{\langle e_k |}{a_k} \left(\frac{\partial f}{a_i \partial u_i} \right) \frac{\partial \, | e_i \rangle}{\partial u_k} \\ &= \sum_{i} \frac{1}{a_i} \frac{\partial}{\partial u_i} \left(\frac{\partial f}{a_i \partial u_i} \right) + \sum_{i,k} \left(\frac{\partial f}{a_i \partial u_i} \right) \frac{\langle a_k\, e_k |}{a_k^2\, a_i} \frac{a_i \partial \, | e_i \rangle}{\partial u_k} \end{aligned}$$

The first sum is already finished. To simplify the second sum we calculate first

$$\begin{aligned} \langle a_k\, e_k | \frac{\partial}{\partial u_k} | a_i\, e_i \rangle &= \langle a_k\, e_k | \frac{a_i \partial \, | e_i \rangle}{\partial u_k} + \langle a_k\, e_k | \, | e_i \rangle \frac{\partial a_i}{\partial u_k} \\ \Rightarrow \quad \langle a_k\, e_k | \frac{a_i \partial \, | e_i \rangle}{\partial u_k} &= \langle a_k\, e_k | \frac{\partial}{\partial u_k} | a_i\, e_i \rangle - \delta_{i\,k}\, a_i \frac{\partial a_i}{\partial u_i} \end{aligned}$$

and secondly (since second order derivatives can be interchanged)

$$\begin{aligned} \langle a_k\, e_k | \frac{\partial}{\partial u_k} | a_i\, e_i \rangle &= \frac{\partial \langle r |}{\partial u_k} \frac{\partial}{\partial u_k} \frac{\partial \, | r \rangle}{\partial u_i} = \frac{\partial \langle r |}{\partial u_k} \frac{\partial}{\partial u_i} \frac{\partial \, | r \rangle}{\partial u_k} \\ &= \frac{1}{2} \frac{\partial}{\partial u_i} \frac{\partial \langle r |}{\partial u_k} \frac{\partial \, | r \rangle}{\partial u_k} = \frac{1}{2} \frac{\partial}{\partial u_i} a_k^2 = a_k \frac{\partial a_k}{\partial u_i} \end{aligned}$$

Combining all equations we finally get

$$\begin{aligned} \Delta f = \text{div grad} f &= \sum_{i} \frac{1}{a_i} \frac{\partial}{\partial u_i} \left(\frac{\partial f}{a_i \partial u_i} \right) + \sum_{i,k} \left(\frac{\partial f}{a_i \partial u_i} \right) \frac{1}{a_k\, a_i} \frac{\partial a_k}{\partial u_i} - \sum_{i} \left(\frac{\partial f}{a_i \partial u_i} \right) \frac{1}{a_i^2} \frac{\partial a_i}{\partial u_i} \\ &= \frac{1}{a_1\, a_2\, a_3} \left[\frac{\partial}{\partial u_1} \left(\frac{a_2\, a_3}{a_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{a_1\, a_3}{a_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{a_1\, a_2}{a_3} \frac{\partial f}{\partial u_3} \right) \right] \end{aligned}$$

thus e.g. for spherical coordinates we get

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2\, \sin\vartheta} \frac{\partial}{\partial \vartheta} \left(\sin\vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{r^2\, \sin^2\vartheta} \frac{\partial^2}{\partial \varphi^2}$$