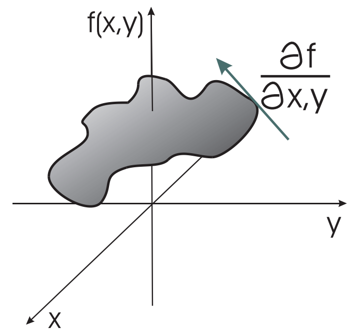
4.5 Total Derivatives

partial derivatives were defined for $f : \mathbb{R}^n \to \mathbb{R}^1$
$\to$ tangents on a hyper surface: We consider now: $f : \mathbb{R}^N \to \mathbb{R}^M$ a vector function.

Definition 40 *$\vec{f} : \mathbb{R}^N \to \mathbb{R}^M$ $\vec{x}, \vec{h} \in \mathbb{R}^N$ $\tilde{A} \in \mathbb{R}^{M\times N} \hat{=} M \times N$ matrix.*
Matrix $\tilde{A}(\vec{x})$ is called the <u>total derivative</u> of $\vec{f}$, if $\vec{f}$ can be expressed as



$$\vec{f}(\vec{x}+\vec{h}) = \vec{f}(\vec{x}) + \tilde{A}\vec{h} + \vec{\varphi}(\vec{h}) \qquad \vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_M \end{pmatrix}$$

$$\textit{where: } \lim_{\vec{h}\to\vec{0}} \frac{\vec{\varphi}(\vec{h})}{|\vec{h}|} = \vec{0} \qquad \vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_M \end{pmatrix}$$

<u>Notes:</u>

- total derivative is a Matrix!!
- if $M = N = 1$ then

$$\begin{aligned} f(x+h) &= f(x) + Ah + \varphi(h) \\ \leftrightarrow \frac{f(x+h)-f(x)}{h} &= A + \frac{\varphi(h)}{h} \\ h \to 0 \Rightarrow A &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x) \Rightarrow \text{ than 1D case o.k.!} \end{aligned}$$

- in components this means:

$$f_j(\vec{x}+\vec{h}) = f_j(\vec{x}) + \underbrace{\sum_{k=1}^{N} a_{jk}h_k}_{\text{Matrix multiplication}} + \varphi_j(\vec{h})$$

- case: $M = 1$:

$$\begin{aligned} f(\vec{x}+\vec{h}) &= f(\vec{x}) + \vec{A}\cdot\vec{h} + \varphi(\vec{h}) \\ \leftrightarrow f(\vec{x}+\vec{h}) - f(\vec{x}) &= \vec{A}\cdot\vec{h} + \varphi(\vec{h}) \\ \vec{h} &= \begin{pmatrix} 0 \\ \vdots \\ h \\ \vdots \\ 0 \end{pmatrix} = \vec{e}_k h \\ \to f(\vec{x}+h\vec{e}_k) - f(\vec{x}) &= A_k h + \varphi(\vec{h}) \quad |: h \\ \frac{f(\vec{x}+h\vec{e}_k) - f(\vec{x})}{h} &= A_k + \frac{\varphi(\vec{h})}{h} \\ \lim_{h\to 0} \frac{f(\vec{x}+h\vec{e}_k) - f(\vec{x})}{h} &= A_k = \frac{\partial f}{\partial x_k} \end{aligned}$$

  <u>Thus:</u>

$$\vec{A} = \vec{\nabla} f : \; f(\vec{x}+\vec{h}) = f(\vec{x}) + \vec{\nabla} f \cdot \vec{h} + \varphi(\vec{h})$$

  is the total derivative $\to$ N-dimensional Taylor expansion up to the first order!
  $\Rightarrow$ Thus, total derivative is the generalization of the direction to the case N, M.

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$$\vec{A}\cdot\vec{h} = \vec{0}$$

  is the parametrization of an M dimensional hyperplane at the point $\vec{h} = \vec{0}$, i.e. at $\vec{x}$ that "touches" the M-D-function

- Note: $\vec{f}:\mathbb{R}^N \to \mathbb{R}^M$. If total derivative exists then
  $\to$ all partial derivatives $\frac{\partial f_k}{\partial x_j}$ of all components with respect to all variables are well defined!

$$\begin{array}{rcl} \text{total derivative} & \Rightarrow & \text{partial derivatives} \\ & \not\Leftarrow & \text{not in general, but for all "friendly" functions o.k.} \\ \text{Hence: total} & \Leftrightarrow & \text{partial} \end{array}$$

Examples

(i) $f:\mathbb{R}^N \to \mathbb{R}$

$$f(\vec{x}) = (x_1,\ldots,x_N)^\top \cdot \tilde{c} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \qquad \tilde{c} - N \times N \text{ Matrix symmetric!}$$

total derivative:

$$\begin{aligned} f(\vec{x}+\vec{h}) - f(\vec{x}) &= (\vec{x}+\vec{h})\cdot\tilde{c}(\vec{x}+\vec{h}) - \vec{x}\cdot\tilde{c}\vec{x} \\ &= (\vec{x}+\vec{h})\cdot(\tilde{c}\vec{x}+\tilde{c}\vec{h}) - \vec{x}\cdot\tilde{c}\vec{x} \\ &= \vec{x}\tilde{c}\vec{x} + \vec{h}\tilde{c}\vec{x} + \vec{x}\tilde{c}\vec{h} + \vec{h}\tilde{c}\vec{h} - \vec{x}\cdot\tilde{c}\vec{x} \\ &= \vec{h}\tilde{c}\vec{x} + \vec{x}\tilde{c}\vec{h} + \vec{h}\tilde{c}\vec{h} \\ &= (\tilde{c}\vec{x})\cdot\vec{h} + (\tilde{c}\vec{x})\cdot\vec{h} + (\tilde{c}\vec{h})\cdot\vec{h} = 2(\tilde{c}\vec{x})\vec{h} + \underbrace{(\tilde{c}\vec{h})\cdot\vec{h}}_{\varphi(\vec{h})} \end{aligned}$$

$$\Rightarrow: \qquad \text{total derivative: } f'(\vec{x}) = 2\tilde{c}\vec{x} = \vec{\nabla} f$$

(ii) $\vec{f}:\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{aligned} \vec{f}(x_1,x_2) &= \begin{pmatrix} x_1 + x_2^2 \\ x_1^2 \end{pmatrix} \qquad \vec{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ \vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) &= \begin{pmatrix} x_1 + h_1 + (x_2+h_2)^2 \\ (x_1+h_1)^2 \end{pmatrix} - \begin{pmatrix} x_1 + x_2^2 \\ x_1^2 \end{pmatrix} \\ &= \begin{pmatrix} h_1 + 2x_2h_2 + h_2^2 \\ 2x_1h_1 + h_1^2 \end{pmatrix} = \tilde{A}\vec{h} + \tilde{\varphi}(\vec{h}) \\ &= \begin{pmatrix} h_1 + 2x_2h_2 \\ 2x_1h_1 + 0\cdot h_2 \end{pmatrix} + \begin{pmatrix} h_2^2 \\ h_1^2 \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ 2x_1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} h_2^2 \\ h_1^2 \end{pmatrix} \\ \frac{\begin{pmatrix} h_2^2 \\ h_1^2 \end{pmatrix}}{|\vec{h}|} &= \begin{pmatrix} \frac{h_2^2}{\sqrt{h_1^2+h_2^2}} \\ \frac{h_1^2}{\sqrt{h_1^2+h_2^2}} \end{pmatrix} \quad \begin{array}{l} 0 < \frac{h_2^2}{\sqrt{h_1^2+h_2^2}} < \frac{h_2^2}{h_2} = h_2 \to 0 \\ 0 < \frac{h_1^2}{\sqrt{h_1^2+h_2^2}} < \frac{h_1^2}{h_1} = h_1 \to 0 \end{array} \quad \frac{\tilde{\varphi}(\vec{h})}{|\vec{h}|} \to 0 \end{aligned}$$

In general:

$$\vec{f}:\mathbb{R}^N \to \mathbb{R}^M \quad \vec{f}(\vec{x}) = \begin{pmatrix} f_1(x_1,\ldots,x_N) \\ f_2(x_1,\ldots,x_N) \\ \vdots \\ f_M(x_1,\ldots,x_N) \end{pmatrix} = \begin{pmatrix} \vdots \\ f_j(\ldots,x_k,\ldots) \\ \vdots \end{pmatrix}$$

than: total derivative is given by the Matrix:

$$\tilde{A} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix} = \left(\frac{\partial f_j}{\partial x_k}\right) \begin{array}{l} j = 1,\ldots,M \\ k = 1,\ldots,N \end{array} \quad M \times N\text{-matrix}$$

$M = 1:\ \tilde{A} = \left(\frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_N}\right)^\top$

$\tilde{A}$ is called Jacobi-Matrix of $\vec{f}$ or the differential of $\vec{f}$ and det $\tilde{A}$ is called the Jacobi or Functional determinant of $\vec{f}$.