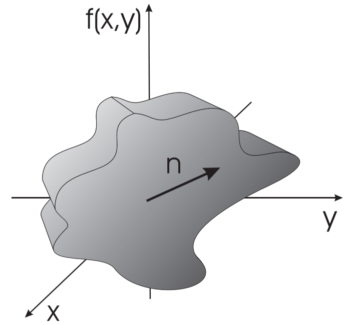
4.4 Derivatives in certain directions

Definition 38 *Derivative in certain direction:*
If $f : \mathbb{R}^n \to \mathbb{R}$*, direction* $\vec{n}$*, than*

$$\frac{\partial f}{\partial \vec{n}} = \lim_{h\to 0} \frac{f(\vec{x}+h\vec{n}) - f(\vec{x})}{h}$$

is defined as the derivative of f *in the direction* $\vec{n}$
$\Rightarrow$ *Thus partial derivative* $\frac{\partial f}{\partial x_k}$ *is derived in the direction* $\vec{e}_k$



Example:

$$\begin{aligned}
f(x_1,x_2) &= x_1^2 + x_2^3 \qquad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\
f(\vec{x}+h\vec{n}) &= (x_1+hn_1)^2 + (x_2+hn_2)^3 \\
f(\vec{x}) &= x_1^2 + x_2^3 \\
\to f(\vec{x}+h\vec{n}) - f(\vec{x}) &= x_1^2 + 2hn_1x_1 + h^2n_1^2 + x_2^3 + 3x_2^2hn_2 + 3x_2h^2n_2^2 + h^3n_2^3 - (x_1^2 + x_2^3) \\
\to \frac{\partial f}{\partial \vec{n}} &= \lim_{h\to 0} \frac{2hn_1x_1 + h^2n_1^2 + 3x_2^2hn_2 + 3x_2h^2n_2^2}{h} = 2n_1x_1 + 3x_2^2n_2 \\
&= \begin{pmatrix} 2x_1 \\ 3x_2^2 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} \cdot \vec{n}
\end{aligned}$$

In general: writing $\vec{n}$ as a linear combination of base vectors, i.e.

$$\vec{n} = \sum n_i \vec{e}_i$$

and taking into account, that differentiation is a linear operation we get

$$\frac{\partial f}{\partial \vec{n}} = \sum \frac{\partial f}{\partial \vec{e}_i} n_i = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{pmatrix}}_{\substack{\text{This vector is called} \\ \text{the gradient of } f}} \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix}$$

Definition 39 $f : \mathbb{R}^N \to \mathbb{R}$ *then gradient is the vector*

$$\vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{pmatrix} \in \mathbb{R}^N$$

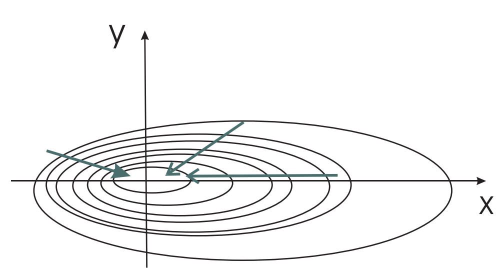
Example:

$$\begin{aligned}
f(\vec{x}) &= \sqrt{x_1^2 + \ldots + x_N^2} \qquad \frac{\partial f}{\partial x_k} = \frac{x_k}{\sqrt{x_1^2 + \ldots + x_N^2}} \\
\to \vec{\nabla} f &= \frac{1}{\sqrt{x_1^2 + \ldots + x_n^2}} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \frac{1}{|\vec{r}|}\vec{r}
\end{aligned}$$

Derivative in a certain direction $\vec{n} \in \mathbb{R}^N$ can be written as:

$$\frac{\partial f}{\partial \vec{n}} = \vec{\nabla} f \cdot \vec{n}$$

Thus as illustrated in the figure, the gradient is the direction of steepest descent.



Higher order partial derivatives:
function: $f : \mathbb{R}^N \to \mathbb{R} \;\; f(x_1, \ldots, x_N)$
partial derivative $\frac{\partial f}{\partial x_k}$ is again a function of $x_1, \ldots, x_N$
second partial derivative:

$$\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_k}\right) = \frac{\partial^2 f}{\partial x_k^2}$$

Also possible:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = \frac{\partial^2 f}{\partial x_j \partial x_k}$$

Always (apart from mathematically pathological cases):

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j} \leftarrow \text{exchange order of derivatives doesn't influence the result}$$

n-th derivative:

$$\frac{\partial^n f}{\underbrace{\partial x_k \ldots \partial x_j}_{n\text{-terms}}}$$

Examples:

(i)

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + x_2^3 \\ \frac{\partial f}{\partial x_1} &= 2x_1 \quad , \quad \frac{\partial f}{\partial x_2} = 3x_2^2 \\ \frac{\partial^2 f}{\partial x_1^2} &= 2 \quad , \quad \frac{\partial^2 f}{\partial x_2^2} = 6x_2 \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} &= 0 = \frac{\partial^2 f}{\partial x_2 \partial x_1} \end{aligned}$$

(ii)

$$\begin{aligned} f(x, y) &= e^{x^2} y^3 \\ \frac{\partial f}{\partial x} &= 2xe^{x^2} y^3 \qquad \frac{\partial f}{\partial y} = 3y^2 e^{x^2} \\ \frac{\partial^2 f}{\partial x^2} &= 2e^{x^2} y^3 (1 + 2x^2) \qquad \frac{\partial^2 f}{\partial y^2} = 6ye^{x^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= 6xy^2 e^{x^2} \qquad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2 e^{x^2} \\ \frac{\partial^3 f}{\partial y \partial x^2} &= 6y^2 e^{x^2} (1 + 2x^2) \end{aligned}$$