4.3 Partial derivatives

Now we consider $f : \mathbb{R}^N \to \mathbb{R}$

Definition 37 *If the limit*

$$\lim_{h\to 0} \frac{f(x_1, x_2, \ldots, x_k + h, \ldots, x_N) - f(x_1, x_2, \ldots, x_k, \ldots, x_N)}{h}$$

exists, then it is called the partial derivative

$$\frac{\partial f}{\partial x_k}$$

of f with respect to x_k.
In vector notation:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \qquad \vec{e_k} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ the 1 is at the k-th position and } \vec{e_k} \in \mathbb{R}^N$$

$$\to \frac{\partial f}{\partial x_k} = \lim_{h\to 0} \frac{f(\vec{x} + h\vec{e_k}) - f(\vec{x})}{h} \quad \text{(note: } \tfrac{df}{dx} \text{ in the case of one variable)}$$

note that we have now round "∂'s "

Example:

$$\begin{aligned} F(x_1, x_2) &= x_1^2 + x_2^3 \\ \frac{\partial f}{\partial x_1} &= \lim_{h\to 0} \frac{(x_1+h)^2 + x_2^3 - (x_1^2 + x_2^3)}{h} = \lim_{h\to 0} \frac{(x_1+h)^2 - x_1^2}{h} \\ &= \lim_{h\to 0} \frac{2x_1 h + h^2}{h} = \lim_{h\to 0}(2x_1 + h) = 2x_1 \\ \text{similar: } \frac{\partial f}{\partial x_2} &= 3x_2^2 \end{aligned}$$

In general:

partial derivative $\frac{\partial f}{\partial x_k}$ $\hat{=}$ "normal" derivative with respect to x_k where the other variables $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N$ are kept constant as parameters!

$\Rightarrow$ partial derivatives totally equal to normal derivatives!

Example:

$$\begin{aligned} f(x,y) &= e^{x^2} y^3 \\ \to \frac{\partial f}{\partial x} &= 2xe^{x^2} y^3 \\ \frac{\partial f}{\partial y} &= 3y^2 e^{x^2} \end{aligned}$$

$$\begin{aligned} f(x_1, \ldots, x_N) &= \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2} = \sqrt{(\vec{r})^2}, \quad \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \\ \to \frac{\partial f}{\partial x_k} &= 2x_k \frac{1}{2}(x_1^2 + \ldots + x_N^2)^{-\frac{1}{2}} = \frac{x_k}{|\vec{r}|} \end{aligned}$$