4.2 Functions of more than one variable

- Functions may depend on more than one variable
  Example:

$$\begin{aligned} f(x,y) &= x^2+y^2 \;\rightarrow \text{ two variables, one function} \\ (x,y)^\top &= \vec{r} \;:\rightarrow\; f(\vec{r}) = (\vec{r})^2 = x^2+y^2 \end{aligned}$$

  "normal" situation in physics e.g. Hamilton function:

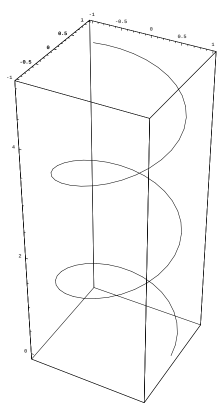
$$H(p,q) = \frac{p^2}{2m} + V(q) \; p\text{-momentum, } q\text{-position}$$

  N variables $x_1,\ldots,x_N \in \mathbb{R}$ vector $\begin{pmatrix} a_1 \\ \vdots \\ x_N \end{pmatrix} = \vec{x}$
  functions $f(x_1,\ldots,x_N) = f(\vec{x}) \;\rightarrow N$ dimensional area in $N+1$ D-space
  2 Variables: $f(x,y)$ area, also niveau-lines

- also possible, function has more than one component→ curve in space
  Example:

$$\vec{f}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix} \quad t \in \mathbb{R} - \text{function itself is a vector, but depends on only one variable}$$

  Other example: spiral in 3D Space



circle in x-y plane (projection)

$$\vec{f}(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ ct \end{pmatrix} \quad \text{a, c, const., } t \in \mathbb{R}$$

  M-Dimensions:

$$\vec{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_M(t) \end{pmatrix} \rightarrow M \text{ dimensional curve.}$$

  Moduls: $|\vec{f}(t)|^2 = f_1^2(t) + \ldots + f_M^2(t)$ but this is not the length of the curve!

General case: N variables and M components

$$\vec{f}: \mathbb{R}^N \rightarrow \mathbb{R}^M \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} \rightarrow \begin{pmatrix} f_1(x_1,\ldots,x_N) \\ f_2(x_1,\ldots,x_N) \\ \vdots \\ f_M(x_1,\ldots,x_N) \end{pmatrix} = \underbrace{\vec{f}}_{M\text{-D-vector}} \Big(\overbrace{\vec{x}}^{N\text{-D-vector}} \Big)$$

Examples:

(i) Combination of circle and straight line $f : \mathbb{R}^2 \quad \Rightarrow \quad \mathbb{R}^3$

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a\cos t_1 \\ a\sin t_1 \\ ct_2 \end{pmatrix} \to \text{complicated function}$$

$$\left.\begin{matrix} a\cos t_1 \\ a\sin t_1 \end{matrix}\right\} \quad \text{circle, and} \quad ct_2 \sim \text{a line}$$

(ii) Electric field as point sources at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} : \vec{E} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{E}(x,y,z) = \frac{q}{4\pi\epsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{q}{4\pi\epsilon_0}\frac{x}{(...)^{\frac{3}{2}}} \\ \frac{q}{4\pi\epsilon_0}\frac{y}{(...)^{\frac{3}{2}}} \\ \frac{q}{4\pi\epsilon_0}\frac{z}{(...)^{\frac{3}{2}}} \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

(iii) Modulus function: $\quad f : \mathbb{R}^N \quad \to \quad \mathbb{R}^1 = \mathbb{R}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad \to \quad \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{\frac{1}{2}} = f(x_1, \ldots, x_N)$$

(iv) Electric field, time dependent:

$$\vec{E}(x,y,z,t) = \begin{pmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \end{pmatrix} \quad \vec{E} : \mathbb{R}^4 \to \mathbb{R}^3$$