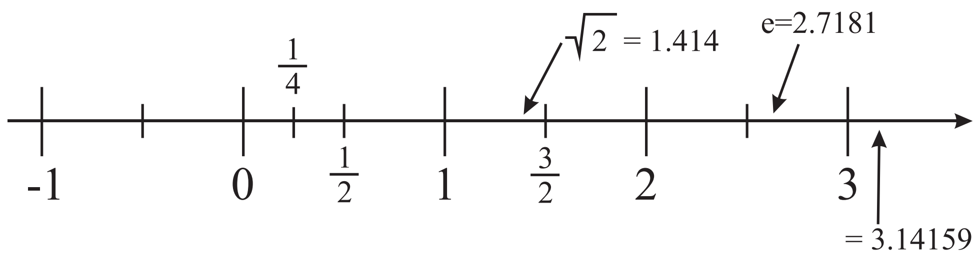
2.1 Complex numbers: Definition

(i) integers: $\{\ldots, -3, -2, -1, \underbrace{0, \underbrace{1, 2, 3, \ldots}_{\mathbb{N}}}_{\mathbb{N}_0}\} = \mathbb{Z}$

(ii) rational numbers: $\mathbb{Q} = \frac{p}{q}$, $p,\ q \in \mathbb{Z};\ \ q \notin 0$

(iii) real numbers: $\mathbb{R}$



$\Longrightarrow$ real numbers are "dense"
many equations have real numbers as solutions e.g.: as solutions e.g.:

$$x^2 - 2 = 0 \quad \Rightarrow \quad x_{1/2} = \pm\sqrt{2} \approx \pm 1.41421\ldots \tag{2.1}$$
$$\sin x = 0 \quad \Rightarrow \quad x = n\pi, \quad n \in \mathbb{Z} \tag{2.2}$$
$$e^x - 10 = 0 \quad \Rightarrow \quad x = \ln 10 \approx 2.302585\ldots \tag{2.3}$$

But:

$$\left.\begin{array}{ll} x^2 + 1 = 0 & \Rightarrow x = ? \\ e^x + 10 = 0 & \Rightarrow x = ? \\ \sin x = 2 & \Rightarrow x = ? \end{array}\right\} \text{no real numbers as solutions!}$$

$\Longrightarrow$ Definition of "new" numbers!

(iv) complex numbers: $\mathbb{C}$
$\Rightarrow$ almost all equations do have solutions in $\mathbb{C}$ (even if they have no real solution)
$\Rightarrow$ simplification of calculations [complex e-function]

One new number is needed:

$$x^2 + 1 = 0 \quad \Leftrightarrow \quad x^2 = -1 \tag{2.4}$$
$$\Leftrightarrow \quad x = \pm\underbrace{\sqrt{-1}}_{i} = \pm i \tag{2.5}$$

Definition 1 *i is a number which square yields -1*
$i^2 = -1$ $\quad\cdot$ *only this number is enough*
$\cdot$ *$i = \sqrt{-1}$ not quite correct!*

$\rightarrow$ all quadratic equations have now (always two!) simple solutions e.g.:

$$x^2 + 2x + 10 \quad = \quad 0 \tag{2.6}$$
$$\rightarrow x_{1/2} \quad = \quad -1 \pm \sqrt{1 - 10} = -1 \pm \sqrt{-9} = -1 \pm \sqrt{-1 \cdot 9} \tag{2.7}$$
$$= \quad -1 \pm 3\sqrt{-1} = -1 \pm 3i \tag{2.8}$$

Test:

$$(-1 + 3i)^2 + 2(-1 + 3i) + 10 = 1 - 9 - 6i - 2 + 6i + 10 = 0$$

same for: $-1 - 3i$
$\Rightarrow$ expressions such as "$-1 + 3i$" are solutions of equations

$\rightarrow$"$-1+3i$" is a new kind of number, a "complex number"
$i =$"imaginary unit"

$$\underbrace{-1}_{\text{real part}} + \overbrace{3}^{\text{imaginary part}} i \quad \Rightarrow \text{A complex number has a real and an imaginary part}$$

$$\left.\begin{array}{lcl} \text{Re}(-1+3i) & = & -1 \\ \text{Im}(-1+3i) & = & 3 \end{array}\right\} -1+3i$$

In general:

Definition 2 $z = a + bi \quad \in \mathbb{C},\ a, b \in \mathbb{R}$
e.g. $z_1 = \sqrt{2} + 5i, \quad z_2 = \frac{5}{3} + \sqrt{3}i$
$\Rightarrow$ *complex numbers are pairs of real numbers with* $i^2 = -1$

$$z = a + bi = (\underbrace{a}_{\text{real}}, \underbrace{b}_{\text{imaginary part}}) = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{2D Vector with two basic vectors}}$$

Definition 3 $\text{Re}\{z\} = a, \quad \text{Im}\{z\} = b$