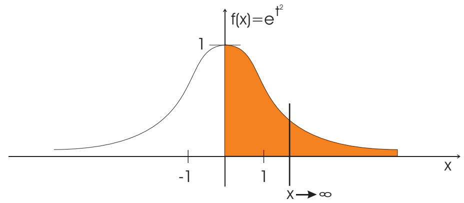
3.19 Important non-elementary functions: Gauss- and Error function

$\int_0^x f(t)dt$ is a function of x "function of the upper limit"
consider: $f(t) = e^{-t^2}$ (Gauß-function or Gaussian)

$$F(x) = \int_0^x e^{-t^2} dt \quad .$$

It is proven that there is <u>no</u> elementary function with $\frac{dF}{dx} = e^{-x^2}$!
However one can show:

$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ (limited)



Definition 35 *The error-function is defined by*

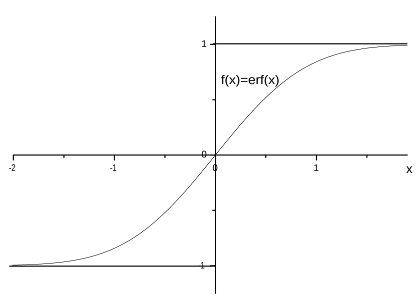
$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

<u>Power series:</u>

$$e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \ldots + \frac{1}{n!}t^{2n} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$$

$$\Rightarrow \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum \frac{(-1)^n}{n!} t^{2n} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} t^{2n+1} = \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3 \cdot 1!} + \frac{t^5}{5 \cdot 2!} - \ldots \right)$$

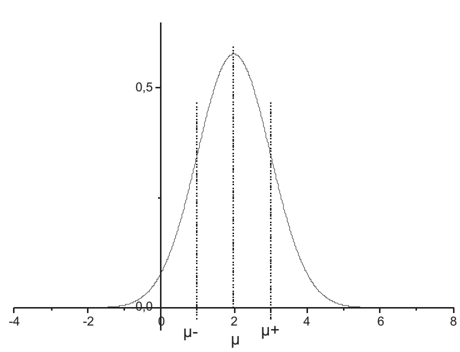
<u>Properties:</u>



$$\begin{aligned} \mathrm{erf}(0) &= 0, \ \mathrm{erf}(x) = -\mathrm{erf}(-x) \\ \mathrm{erf}(\infty) &= \lim_{x\to\infty} \mathrm{erf}(x) = 1 \\ \mathrm{erf}'(x) &= e^{-x^2} \frac{2}{\sqrt{\pi}} \\ \mathrm{erf}''(x) &= -\frac{4}{\sqrt{\pi}} x e^{-x^2} \to x_0 = 0 \ \text{ only inflection point.} \end{aligned}$$

The error-function is essential in probability theory:

$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



$\hat{=}$ normal distribution of a random quantity characterized by two parameter: μ, σ
Normalization:

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$$

We prove this relation by calculating the square of the integral

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2}e^{-y^2}dx\,dy \\
&= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\,r\,dr\,d\varphi \quad \text{(using zylindrical coordinates)} \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} e^{-z}\,dz\,d\varphi \quad \text{(substitution: } z = r^2\text{)} \\
&= -\pi \quad e^{-z}\Big|_0^{\infty} = \pi
\end{aligned}$$

μ : mean value of x

$$\int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\mu e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \mu$$

σ : variance, standard deviation of x from the mean value μ

$$\begin{aligned}
\int_{-\infty}^{+\infty}(x-\mu)^2 f(x)dx &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\sigma^2(x-\mu)\frac{(x-\mu)}{\sigma^2}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\left\{-\sigma^2(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx\right\} \\
&= 0 + \sigma^2\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \\
&= \sigma^2
\end{aligned}$$

~67% of all values of the random variable x are with the interval $[\mu-\sigma, \mu+\sigma]$
~95% are in $[\mu - 2\sigma, \mu + 2\sigma]$
$f(x)$ probability density $\rightarrow$ probability?
The probability that a value x in the interval $[a, b]$ is given by:

$$\begin{aligned}
P(a \le x \le b) &= \frac{1}{\sqrt{2\pi}\sigma}\int_a^b e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt \\
\text{defining: } \Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt = \frac{1}{2}\left[\mathrm{erf}(\frac{x}{\sqrt{2}}) + 1\right] \\
\text{we get: } P(a \le x \le b) &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)
\end{aligned}$$