3.18 Important non-elementary functions: Delta function

The δ function for real numbers x corresponds to the Kronecker δ_{ij} for integer numbers i, j. It is defined by

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \text{and} \qquad \int_{-\epsilon}^{+\epsilon} \delta(x) dx = 1$$

The major property of the δ function is it's projection property within an integral for all reasonable functions $f(x)$

$$f(x_0) = \int_{x_0-\epsilon}^{x_0+\epsilon} f(x)\delta(x - x_0) dx$$

Exactly spoken, the δ function is not a function but a distribution, i.e. there are many representation of this function. All of them need a limiting process. Some important representation of the δ function are

$$\delta(x) = \lim_{\epsilon \to 0} \begin{cases} 0 & |x| > \frac{\epsilon}{2} \\ \frac{1}{\epsilon} & |x| \leq \frac{\epsilon}{2} \end{cases}$$

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

For basis vectors e_n with integer index n the Kronecker δ is used to define the orthonormality relation

$$\langle e_n | e_m \rangle = \delta_{n,m}$$

One very important example is

$$\delta_{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik(n-m)} dk$$

For basis vectors $f_k(x)$ with continuous index k orthonormality is defined by the δ function

$$\langle f_k(x) | f_l(x) \rangle = \int f_k^*(x) f_l(x) dx = \delta(k - l)$$

e.g. for the Fourier functions we have

$$\left\langle e^{ikx} | e^{ilx} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} e^{ilx} dx = \delta(k - l)$$