3.17 Important non-elementary functions: Gamma function

Factorial function: $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$
Thus:

$$\begin{aligned} n! &= n \cdot (n-1)! \qquad \text{and definition for convenience} \\ (n+1)! &= (n+1)n! \qquad 0! = 1 \\ n! &= n(n-1)! = n(n-1) \cdot (n-2)! \end{aligned}$$

only defined for natural numbers $n \geq 0$.
This will now be generalized for real values.
Consider:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad \text{for } x > 0$$

Than it is simple to show that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, 3, \ldots$

$$\left(\int_0^\infty e^{-t} t^n dt = n! \right)$$

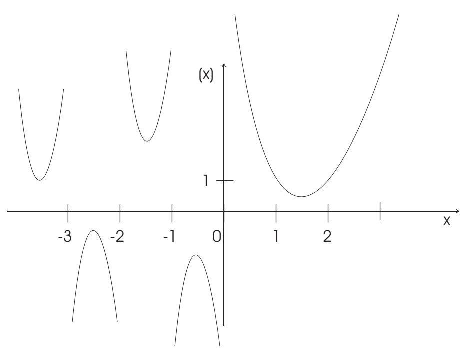
$\Rightarrow$ Function $\Gamma(n)$ is equivalent to $(n-1)!$ for natural numbers n.

$$\begin{aligned} \Gamma(x+1) = \int_0^\infty e^{-t} t^x dt &= -e^{-t}\ t^x|_0^\infty + \int_0^\infty e^{-t} x t^{x-1} dt = x\Gamma(x) \\ \text{Thus: } \Gamma(x+1) = x\Gamma(x) &= x \cdot (x-1)\Gamma(x-1) = x(x-1)(x-2)\Gamma(x-2) = ... \end{aligned}$$

Same property as factorial but not restricted to integer numbers. The Γ function exists for real numbers, except for negative integers and zero, since

$$\Gamma(x+1) = x\Gamma(x) \quad \Rightarrow \quad 1 = 0! = \Gamma(1) = 0\Gamma(0),$$

i.e $\Gamma(0) = \infty$ and thus $\Gamma(n) = \infty$ for negative integers.



Special values:

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} \frac{1}{\sqrt{t}} dt = 2 \int_0^\infty e^{-t^2} dt = \sqrt{\pi}$$

Stirling Formula: An asymptotic approximation for the factorial function one gets by calculating the Taylor approximation up to quadratic order of

$$g(x) = \ln\left(x^n \exp(-x)\right) = n \ln x - x \qquad ,$$

i.e.

$$\frac{dg}{dx} = \frac{n}{x} - 1 \qquad \text{and} \qquad \frac{d^2g}{dx^2} = -\frac{n}{x^2} < 0$$

around it's maximum at $x_0 = n$.
We find

$$g(x) \approx g(x_0) + \frac{dg}{dx}(x_0)(x - x_0) + \frac{1}{2}\frac{d^2g}{dx^2}(x_0)(x - x_0)^2 = n \ln n - \frac{(x-n)^2}{2n}$$

finally giving

$$n! = \int_0^\infty \exp\left(g(x)\right) dx \approx \int_0^\infty \exp\left(n\ln(n) - \frac{(x-n)^2}{2n}\right) dx = \left(\frac{n}{e}\right)^n \int_0^\infty \exp\left(-\frac{(x-n)^2}{2n}\right) dx$$

For large n ($n > 3$)

$$\int_0^\infty \exp\left(-\frac{(x-n)^2}{2n}\right) dx \approx \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2n}\right) dx = \sqrt{2\pi n}$$

A slightly improved approximation for large n gives

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1 + \frac{1}{12n}\right)$$