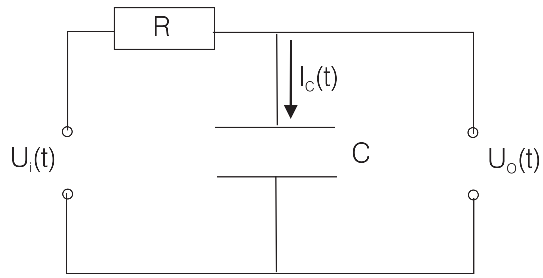
3.16 Fourier Transformation: Solving DEQs

Example 1: Low pass filter:

We know:

$$
\begin{aligned}
U(t) &= R\,I(t) \quad \text{(resistor)} \\
U(t) &= \frac{Q(t)}{C} \quad \text{(capacitor)}.
\end{aligned}
$$



So according to the Kirchhoff equations we find for the current across the capacitor I_C

$$\frac{U_i(t) - U_o(t)}{R} = I_C(t) = C\dot{U}_o \quad .$$

After Fourier-Transformation this differential equation for U_o translates into

$$\frac{U_i(\omega) - U_o(\omega)}{R} = i\omega C U_o(\omega) \quad ,$$

i.e.

$$U_o(\omega) = \frac{U_i(\omega)}{1 + i\omega RC} := A(\omega) U_i(\omega) \quad ,$$

which is the response to a sinusoidally modulated input signal with frequency $\omega/2\pi$. Using the example IV for calculation of Fourier transformed we can find the original function $A(t)$ of $A(\omega)$

$$
\begin{aligned}
A(t) &= \mathcal{F}^{-1}\left\{\frac{1}{1 + i\omega RC}\right\} \\
&= \frac{\sqrt{2\pi}}{RC}\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\frac{1}{RC} + i\omega}\right\} = \left\{\begin{array}{ll} \frac{\sqrt{2\pi}}{RC} e^{-\frac{t}{RC}} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{array}\right.
\end{aligned}
$$

According to the convolution theorem we find

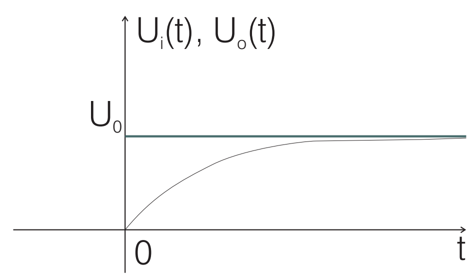
$$
\begin{aligned}
U_o(t) &= \mathcal{F}^{-1}\left\{\frac{1}{1 + i\omega RC} U_e(\omega)\right\} \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} A(t-\tau)\, U_i(\tau)\, d\tau = \frac{1}{RC} \int\limits_{-\infty}^{t} e^{-\frac{t-\tau}{RC}}\, U_i(\tau)\, d\tau
\end{aligned}
$$

So this integral allows to calculate the response $U_o(t)$ of a low pass filter to an arbitrary input signal $U_i(t)$. Typically a filter is characterized in real space by the response to a step like perturbation

$$U_i(t) = \left\{\begin{array}{ll} U_0 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{array}\right. \quad ,$$

which for our case is

$$
\begin{aligned}
U_o(t) &= \frac{1}{RC} \int\limits_{0}^{t} e^{-\frac{t-\tau}{RC}}\, U_0\, d\tau \\
&= U_0 \left(1 - e^{-\frac{t}{RC}}\right)
\end{aligned}
$$



In this example we calculated the two typical descriptions of a filter:

- the transfer function $A(\omega)$ in Fourier space
- the answer to a step like perturbation in real space, i.e. the transient.