3.15 Fourier Transformation: Properties

(i) Linearity:

$$f(x), g(x), F(p), G(p), \quad a, b \in \mathbb{R}$$

$$\begin{aligned} \mathcal{F}\{af(x)+bg(x)\} &= a\mathcal{F}\{f(x)\}+b\mathcal{F}\{g(x)\} \\ &= aF(p)+bG(p) \end{aligned}$$

(ii) Differential Properties: $f(x)$ and derivative: $f'(x) = \frac{df}{dx}$

$$\begin{aligned} \mathcal{F}\{f'(x)\} &= ip\mathcal{F}\{f(x)\} = ipF(p) \\ f^n(x) &= \frac{d^n f}{dx^n} \text{ n-th derivative: } \mathcal{F}\{f^n(x)\} = (ip)^n F(p) \\ &\Rightarrow \text{ derivative leads to a factor } ip \end{aligned}$$

(iii) Convolution theorem:

$$\begin{aligned} \mathcal{F}^{-1}\left\{\sqrt{2\pi}F(p)\cdot G(p)\right\} &= \int_{-\infty}^{+\infty} f(x-t)g(t)dt = f\star g(x) \\ \text{and } \mathcal{F}\left\{\sqrt{2\pi}f(x)g(x)\right\} &= F\star G(p) = \int_{-\infty}^{+\infty} F(p-p')G(p')dp' \end{aligned}$$

Definition 34 *Convolution $f\star g$*

$$\begin{aligned} f\star g(x) &:= \int_{-\infty}^{+\infty} f(x-t)g(t)dt \\ & \quad \textit{t has negative and positive values ($\pm$direction)} \\ f\star g(x) &= g\star f(x) \end{aligned}$$