3.14 Fourier-Transformation: Definition

We define:

$$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ipx} dx$$

Note: The function $f(x)$ does not have to be periodic!

For $\lim_{x\to\infty}$ the function $f(x) \to 0$ must be "fast enough", otherwise the integral can not be calculated, i.e. the Fourier-transform does not exist. For such functions a Laplace-transformation often can replace the Fourier-transformation, but this will not be discussed in this lecture.

$F(p)$ is called the Fourier-transform of $f(x)$.

The back-transformation is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(p) e^{+ipx} dp$$

Remarks:

- $f(x) \Leftrightarrow F(p)$ unitary transformation
- we write in the following:

$$\begin{aligned} F(p) &= \mathcal{F}\{f(x)\} \\ f(x) &= \mathcal{F}^{-1}\{F(p)\} \end{aligned}$$

- FT is the generalization of Fourier-series for $L \to \infty, n \to \infty, p_n \to p$ cont. than $p_n \to p, \sum_n \to \int \dots dp$
- Delta function: $F(p) = 1 \to f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ipx} dx = \delta(x) \to$ strange function $\to$ later!
- factor $\frac{1}{\sqrt{2\pi}}$ sometimes different: you will also find:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(p) e^{ipx} dp \quad , \quad F(p) = \int_{-\infty}^{+\infty} f(x) e^{-ipx} dx$$

- higher dimensions: $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$: Fourier-transform

$$\begin{aligned} \vec{x} &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \\ F(\vec{p}) &= \frac{1}{(\sqrt{2\pi})^n} \iint_{-\infty}^{+\infty} \dots \int f(\vec{x}) e^{-i\vec{p}\cdot\vec{x}} dx_1 dx_2 \dots dx_n \\ f(\vec{x}) &= \frac{1}{(\sqrt{2\pi})^n} \iint_{-\infty}^{+\infty} \dots \int f(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} dp_1 dp_2 \dots dp_n \end{aligned}$$