3.13 From Fourier series to Fourier-Transformation

We already states that functions with periodicity length L can be developed into a Fourier series:

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{L}nx} \quad \text{for period } L \\
c_n &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-i\frac{2\pi}{L}nx}dx
\end{aligned}
$$

Non periodic functions can be interpreted as functions with periodicity length $L \to \infty$.
We introduce a new variable $k = \frac{2\pi}{L}n$, i.e. $\Delta k = \frac{2\pi}{L}$; so k becomes a contiuous variable for $L \to \infty$. Replacing n by k in the above equations we get

$$
\begin{aligned}
f(x) &= \frac{L}{2\pi}\sum_{n=-\infty}^{\infty} c_k e^{ikx}\Delta k \quad \text{for period } L \\
c_k &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-ikx}dx
\end{aligned}
$$

The decisive step now is to shift the variable L from the first equation into the second equation which as we will see in consequence translates the Kronecker-δ into the δ function:

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(k)e^{ik(n)x}\Delta k \quad \text{for period } L \\
F(k) &= \frac{L}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-ikx}dx
\end{aligned}
$$

Now the transition $L \to \infty$ is possible, leading to

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk \quad \text{for non periodic functions} \\
F(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx
\end{aligned}
$$

Various versions of this equations exist, since the minus sign in the exponent can be shifted from the second into the first equation (, i.e. replacing k by $-k$) and the scaling factor $\frac{1}{2\pi}$ can be shifted to the second equation or in a symmetrical definition the square root of this factor can be written in front of both integrals.