3.12 Fourier series in complex description

$$\cos x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

Thus: $$f(x) = \frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos(kx)+b_k\sin(kx)\right) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

please beware of the negative indices

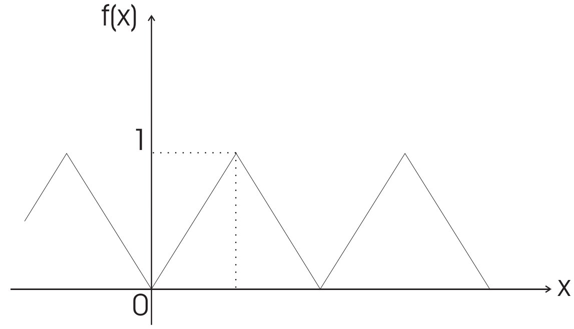
with: $c_0 = \frac{a_0}{2}$ and $c_k = \frac{1}{2}(a_k - ib_k)$; $c_{-k} = \frac{1}{2}(a_k+ib_k)$

and $c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{ikx}dx \left(=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx \mp \frac{1}{\pi}i\int_0^{2\pi} f(x)\sin(kx)dx\right)$

for $k = 0, \pm1, \pm2, \pm3, \dots$

c_k are the complex Fourier coefficients of $f(x) \Rightarrow$ more compact treatment possible. As example:

$$f(x) = \begin{cases} \frac{x}{\pi} & \text{for } 0 \le x < \pi \\ \frac{2\pi - x}{\pi} & \text{for } \pi \le x < 2\pi \end{cases} \quad \text{periodic continuation}$$



$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\left(\int_0^{\pi}\frac{x}{\pi}e^{-ikx}dx + \int_{\pi}^{2\pi}\frac{2\pi-x}{\pi}e^{-ikx}dx\right)$$

Note: $$\int xe^{-ikx}dx = \frac{1}{k^2}e^{-ikx}(ikx+1)+C = \frac{1}{2}x^2+C$$

$$\rightarrow c_0 = \frac{1}{2\pi^2}\left(\frac{1}{2}\pi^2-\frac{1}{2}0^2+2\pi\pi-\left(\frac{1}{2}(2\pi)^2-\frac{1}{2}\pi^2\right)\right) = \frac{1}{2\pi^2}\pi^2 = \frac{1}{2}$$

$$c_k = \frac{1}{2\pi^2}\left[\frac{1}{k^2}e^{-ik\pi}(ik\pi+1)-\frac{1}{k^2}-\frac{1}{k^2}\overbrace{e^{-i2\pi k}}^{1}(ik2\pi+1)+\frac{1}{k^2}e^{-ik\pi}(ik\pi+1)\right.$$
$$\left.+\frac{2\pi}{-ik}\left(e^{-ik2\pi}-e^{-ik\pi}\right)\right]$$
$$= \frac{1}{2\pi^2 k}\left[e^{-ik\pi}(i\pi+i\pi-2\pi i)+\frac{1}{k}e^{-ik\pi}(1+1)-\frac{2}{k}+(-2\pi i+2\pi i)\right]$$
$$= \frac{2}{2\pi^2k^2}\left((-1)^k-1\right) = \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{2}{\pi^2k^2} & \text{if } k \text{ odd}\end{cases}$$

$$\rightarrow f(x) = \frac{1}{2}-\frac{2}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{(2k+1)^2}e^{i(2k+1)x} = \frac{1}{2}-\frac{4}{\pi^2}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\dots\right)$$

$$x=0 \quad\rightarrow\quad f(0) = 0 = \frac{1}{2}-\frac{4}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right) \qquad \text{Thus:} \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$\Rightarrow$ Fourier series be used to calculate series in the above manner!

Example is $\sum_{k=1}^{\infty}\frac{1}{k^6} = \frac{\pi^6}{945}$ (Euler 18th century)

Major application of Fourier series: Arbitrary periodic function→ $\sin, \cos$!

<u>In general:</u>

$$f(x) = \sum_{n=-\infty}^{\infty} c_k e^{i\frac{2\pi}{L}nx} \quad \text{for period } L \quad c_n = \frac{1}{L}\int_0^L f(x)e^{-i\frac{2\pi}{L}nx}dx$$