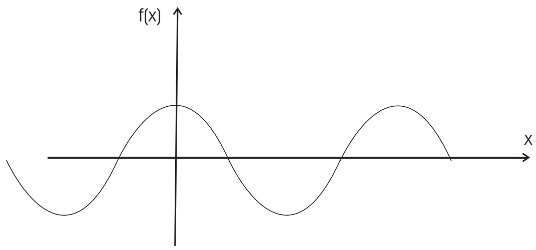
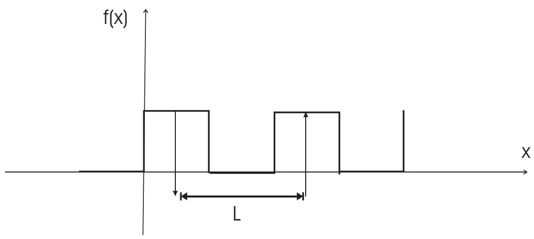
3.11 Fourier series

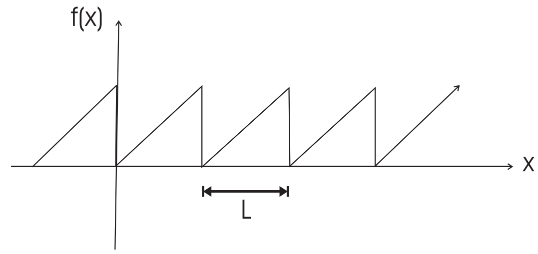
Periodic functions:



$f(x) = \sin x$ o.k.

<u>But also:</u>





Definition 33 *A function $f(x)$ is called periodic if $L > 0$ exists with $f(x) = f(x+L)$ for all x. L is called the periodic of f.*

<u>Example:</u>

$$\begin{aligned} f(x) &= \sin x \qquad L = 2\pi \\ f(x) &= \sin^2 x \qquad L = \pi, \text{ also: } f(x) = f(x+nL),\ n \in \mathbb{Z} \\ f(x) &= \sin 10x \ \rightarrow \quad L = \frac{2\pi}{10} \end{aligned}$$

$\Rightarrow$ now we will see that $\sin x$ and $\cos x$ are prototypes of periodic functions.

If f has period L than $F(x) = f(\frac{L}{2\pi}x)$ has the period 2π
<u>Example:</u>

$$f(x) = \sin(10x) \ \rightarrow \ f(\frac{2\pi}{10*2\pi}x) = \sin x \ \text{ period } 2\pi$$

$\Rightarrow$ | It is sufficient to investigate functions with period 2π. Transformation $\frac{2\pi}{L}x$ will give arbitrary period, i.e. $f(x) = F(\frac{2\pi}{L}L)$

$$\left.\begin{aligned} \frac{1}{\pi}\int_0^{2\pi} \sin(kx)\sin(nx)dx &= \delta_{kn} \\ \frac{1}{\pi}\int_0^{2\pi} \cos(kx)\cos(nx)dx &= \delta_{kn} \\ \frac{1}{\pi}\int_0^{2\pi} \cos(0x)\cos(0x)dx &= 2 \\ \frac{1}{\pi}\int_0^{2\pi} \sin(kx)\cos(nx)dx &= 0 \end{aligned}\right\} k, n \in \mathbb{N}$$

important formulas, which show that sin and cos can be used as a "base" in a vector space

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \\ \text{with: } a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \quad (2 \times \text{mean over period}), \\ a_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx, \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx, \end{aligned}$$

is called the Fourier-Series of a function $f(x)$ with period 2π; a_k and b_k are called the Fourier coefficients of $f(x)$ Exploiting the symmetries of function one can considerably reduce the mathematical effort for calculation Fourier coefficients:
since the periodicity length is 2π $f(-\pi + x) = f(\pi + x)$; so the limits in the integrals for the Fourier coefficients can always be changed to a symmetrical representation, i.e.

$$\int_0^{2\pi} ...dx = \int_{-\pi}^{\pi} ...dx$$

Thus:

$$\begin{aligned}
f(x) &= -f(-x) \rightarrow \text{anti-symmetric} && \rightarrow \quad \text{cos-coeff. are zero!} \quad \int_{-\pi}^{\pi} f(x)\cos(kx)dx = 0 \\
& && \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin(kx)dx = 2\int_0^{\pi} f(x)\sin(kx)dx \\
f(x) &= f(-x) \rightarrow \text{symmetric} && \rightarrow \quad \text{sin-coeff. are zero!} \quad \int_{-\pi}^{\pi} f(x)\sin(kx)dx = 0 \\
& && \text{and} \quad \int_{-\pi}^{\pi} f(x)\cos(kx)dx = 2\int_0^{\pi} f(x)\cos(kx)dx
\end{aligned}$$