3.6 Taylor series and error estimation

If $f(x)$ is a "normal" function than

$$f(x) = f(x_0) + \frac{1}{1!}(x-x_0)f'(x_0) + \frac{1}{2!}(x-x_0)f''(x_0) + \ldots + \frac{(x-x_0)^{n-1}}{(n-1)!}f^{n-1}(x_0) + \frac{(x-x_0)^n}{n!}f^n(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!}f^{n+1}(x_0+\epsilon\Delta x)$$

i.e. the error term

$$\Delta f = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{n+1}(x_0+\epsilon\Delta x) \qquad \begin{array}{rcl}\Delta x & = & x - x_0 \\ 0 & < & \varepsilon < 1\end{array}$$

is the difference between the exact function and the approximation.

<u>Example:</u>

$$\begin{array}{rcl} f(x) & = & \sin x; \qquad x_0 = 0 \\ \text{Taylor: } f(x) & = & x - \frac{x^3}{3!} + \frac{x^5}{5!}f^5(\epsilon x) \qquad 0 < \epsilon < 1 \\ f^5(x) & = & \cos x \Rightarrow \left| f(x) - \left(x - \frac{x^3}{3!}\right)\right| = |\Delta f| = \frac{x^5}{5!}|\cos(\epsilon x)| \le \frac{x^5}{5!} \rightarrow \text{may diverge for large } x \\ \text{but for: } x & = & 10^\circ = \underbrace{\frac{10\pi}{180}}_{0.17} \rightarrow \left| f(x) - (x - \frac{x^3}{3!})\right| \le 10^{-6} \end{array}$$

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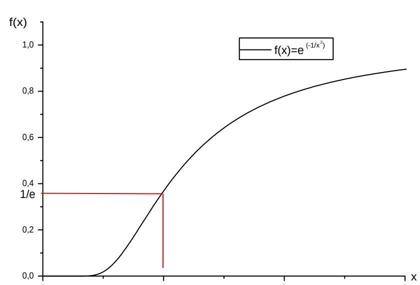
$$\sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

is called a Taylor approximation or series of the order n to the function $f(x)$ at the point x_0.

<u>Example:</u> Approximation of third order to $f(x) = e^x$ at $x_0 = 0$:

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \mathrm{O}(x^4)$$

notation "zero of x^4" means approximately up to x^3 exact
<u>Sometimes problems: (rare cases)</u>



$f(x) = e^{-\frac{1}{x^2}}$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. $\rightarrow$ well defined function.

Taylor-Series around $x_0 = 0$

$$\begin{array}{rcl} f'(x) & = & \frac{2}{x^3}e^{-\frac{1}{x^2}}x + 0; \quad x = 0: \quad \frac{f(x)-f(0)}{x-0} = \frac{e^{-\frac{1}{x^2}}}{x} \\ \frac{1}{x} & = & m \rightarrow \frac{f(x)-f(0)}{x-0} = \frac{e^{-m^2}}{\frac{1}{m}} = me^{-m^2} \rightarrow 0 \quad \text{for } m \rightarrow \infty \\ & \Rightarrow & f'(0) = 0 \quad \text{and also } f^n(0) = 0 \quad \text{for all } n!! \\ & \Rightarrow & \text{Thus Taylor-series of } f(x) \text{ is } \equiv 0 \Rightarrow \text{unsuccessful approximation!!} \end{array}$$