3.4 Sequences and Series

Sequences: $\{a_n\}$ $n = 0, 1, 2, \ldots$ $a_n \in \mathbb{R}$ or $\mathbb{C}$ is called a sequence

$$\{a_n\} \overset{n\to\infty}{\longrightarrow} a \quad \text{or} \quad \lim_{n\to\infty} a_n = a \quad \text{if the sequence converges to a number } a \in \mathbb{R}$$

Series: $\{a_n\}$ sequence then $\sum_{n=0}^{\infty} a_n$ is called an infinite series

$$\begin{array}{ll} \sum_{n=0}^{\infty} a_n & \text{infinite series?} \\ \sum_{n=0}^{N} a_n & \text{finite series = o.k.} \end{array}$$

If $S_N = \sum\limits_{n=0}^{\infty} a_n$ converges then this means that $\lim\limits_{N\to\infty} S_N = \lim\limits_{N\to\infty} \sum\limits_{n=0}^{N} a_n = \sum\limits_{n=0}^{\infty} a_n = \alpha$

condition: $a_n \to 0$ if $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty!$, hence $a_n \to 0$ is not a sufficient condition for a series to converge.

Nearly all theoretical considerations related to the convergence conditions for series rely on the geometrical series:

$$\sum_{n=0}^{\infty} x^n; \qquad a_n = x^n$$

We can use a small trick to calculate the sum

$$S_N = \sum_{n=0}^{N} x^n.$$

Let us write the sum twice

$$\begin{array}{rclllll} S_N & = & 1+ & x+ & x^2+ & \ldots+ & x^N \\ x\,S_N & = & & x+ & x^2+ & \ldots+ & x^N+ \quad x^{N+1} \end{array}$$

subtracting both lines we get

$$(1-x)\,S_N = 1 - x^{N+1} \quad \text{or} \quad S_N = \frac{1-x^{N+1}}{1-x}$$

so for $|x| < 1$ we find

$$\lim_{N\to\infty} S_N = \frac{1}{1-x}$$

and for $|x|\rangle 1$ the series diverges.

For the geometric series we have two conditions for the components a_n

(i)
$$\frac{a_{n+1}}{a_n} = x$$

(ii)
$$\sqrt[n]{a_n} = x$$

These properties allow to specify the conditions to take the geometric series as a majorant for other series leading to the two standard convergence criteria for infinite series:

(i) If $\sum\limits_{n=0}^{\infty} a_n$ is given and q, n_0 exist with $\left|\frac{a_{n+1}}{a_n}\right| \leq q < 1$ for $n \geq n_0$ then $\sum\limits_{n=0}^{\infty} a_n$ is convergent.
If $\left|\frac{a_{n+1}}{a_n}\right| \geq q > 1$ then $\sum a_n$ is divergent, if $\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow 1$ no decision is possible.

(ii) If q, n_0 exist with $\sqrt[n]{a_n} \leq q < 1$ for $n \geq n_0$ then $\sum a_n$ convergent. If $\sqrt[n]{a_n} \geq q \geq 1$ then $\sum a_n$ divergent. If $\sqrt[n]{a_n} \to 1$ no decision is possible.