## 3.3 Calculation rules for integrals

Nearly all integrals are solved by applying the fundamental theorem of calculus:

$$\begin{aligned} \int_a^b f(x)dx &= F(b) - F(a) \quad \text{where } f(x) = \tfrac{dF}{dx} \\ &= [F(x)]_a^b \end{aligned}$$

$\Rightarrow$ Calculation of integrals $\Leftrightarrow$ Finding function $F(x)$
A function $F$ with the property $f(x) = \frac{dF}{dx}$ is called primitive with respect to $f(x)$.
Plus two rules:

- integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

which directly follows from the product rule for calculating derivatives.

- substitution:

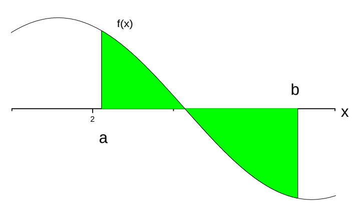
$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

which directly follows from the chain rule for calculation derivatives.

Integrals: basic rules

$$\begin{aligned} \int_a^b f(x)dx &= -\int_b^a f(x)dx \;\rightarrow\; \text{area with sign!} \\ \int_a^a f(x)dx &= 0 \\ \int_a^b [f(x) \pm g(x)]\, dx &= \int_a^b f(x)dx \pm \int_a^b g(x)dx \\ \int_a^b kf(x)dx &= k\int_a^b f(x)dx \quad \Rightarrow \text{"vector space"} \\ \int_a^b f(x)dx &= 0 \not\Rightarrow f(x) \equiv 0 \end{aligned}$$

because




Indefinite integral:

$$\int_{x_0}^{x} f(y)dy = F(x) - F(x_0)$$

$$\rightarrow F(x) = \int_{x_0}^{x} f(y)dy + C$$

$$\text{or shorter } F(x) = \int f(x)dx + C \leftarrow \text{Indefinite integral, means } \int_{x_0}^{x} f(y)dy!!$$

$\Rightarrow$ basic integrals:

$$\begin{aligned}
\int x^n dx &= \frac{1}{n+1} x^{n+1} + C, n \neq -1\, n \in \mathbb{Z} \\
\int x^r dx &= \frac{1}{r+1} x^{r+1} + C,\ r \in \mathbb{R}\,\text{but}\ r \neq -1 \\
\int \frac{1}{x} dx &= \ln|x| + C \\
\int e^x dx &= e^x + C \\
\int \sin x dx &= -\cos x + C \\
\int \cos x dx &= \sin x + C
\end{aligned}$$

Example for substitution rule:

$$\int_0^1 x e^{x^2} dx = ?$$

$$\begin{aligned}
g(x) &= x^2 \rightarrow & g'(x) &= 2x \\
f(g) &= e^g \rightarrow & F(g) &= e^g
\end{aligned}$$

$$\begin{aligned}
\rightarrow \int x e^{x^2} dx &= \int \tfrac{1}{2} g'(x) e^{g(x)} dx \\
&= \tfrac{1}{2} e^{x^2} \\
\rightarrow \int_0^1 x e^{x^2} dx &= \left[\tfrac{1}{2} e^{x^2}\right]_0^1 = \tfrac{1}{2}(e-1)
\end{aligned}$$