3.2 Calculation rules for derivatives

Derivatives: Using binomial coefficients (see exercise 1) we can write for $n \in \mathbb{N}$

$$f(x) = x^n; \qquad f(x+h) = (x+h)^n = \binom{n}{0} x^n + \binom{n}{1} h x^{n-1} + \binom{n}{2} h^2 x^{n-2} + ...$$

leading to

$$\frac{f(x+h)-f(x)}{h} = n\, x^{n-1} + \binom{n}{2} h x^{n-2} + ... \to n x^{n-1} \quad \text{if} \quad h \to 0$$

i.e.

$$f(x) = x^n \to n x^{n-1} = f'(x) \quad n \in \mathbb{N}$$

This already allows to calculate the derivatives of all functions which are defined by Taylor series.
Plus:

$$\begin{array}{rcl} [f(x) \pm g(x)]' &=& f'(x) \pm g'(x) \\ [k f(x)]' &=& k f'(x) \end{array} \Rightarrow \text{"functions form vector-space"}$$

Plus:

$$\begin{array}{lrcl} \text{product rule:} & [f(x)g(x)]' &=& f'(x)g(x) + f(x)g'(x) \\ \text{quotient rule:} & \left[\frac{f(x)}{g(x)}\right]' &=& \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ \text{chain rule:} & [f(g(x))]' &=& g'(x) f'(g(x)) \end{array}$$

To calculate the derivatives of inverse functions f^{-1} which are defined by

$$x = f(f^{-1}(x))$$

we can apply the chain rule:

$$1 = \frac{d}{dx} f(f^{-1}(x)) = \frac{df}{dx}\left(f^{-1}(x)\right) * \frac{df^{-1}}{dx}(x).$$

For the natural logarithm which is the inverse of the exponential function, i.e.

$$x = e^{\ln(x)}$$

we find

$$1 = \frac{de^x}{dx}(\ln(x)) * \frac{d\ln(x)}{dx}(x) = x\, \frac{d\ln(x)}{dx},$$

i.e.

$$\frac{d\ln(x)}{dx} = \frac{1}{x}.$$

This now allows to calculate the derivative of $f(x) = x^r$ with $r \in \mathbb{R}$

$$\frac{dx^r}{dx} = \frac{de^{r\ln(x)}}{dx} = \frac{de^x}{dx}(r\ln(x)) * \frac{d\,(r\ln(x))}{dx} = x^r \frac{r}{x} = r x^{r-1}$$

in summary:

$$\left.\begin{array}{rclcrcl} f(x) &=& x^r & \to & f'(x) &=& r x^{r-1}, \ \ r \in \mathbb{R} \\ f(x) &=& e^x & \to & f'(x) &=& e^x \\ f(x) &=& \ln|x| & \to & f'(x) &=& \frac{1}{x} \ \ !! \\ f(x) &=& \sin x & \to & f'(x) &=& \cos x \\ f(x) &=& \cos x & \to & f'(x) &=& -\sin x \end{array}\right\} \Rightarrow \text{all elementary functions!}$$

Examples:

(i)

$$\begin{array}{rcl} f(x) = \sin(x \cdot \ln x) \Rightarrow f'(x) &=& (x \ln x)' \cos(x \ln x) \\ &=& (1 \ln x + x\frac{1}{x}) \cos(x \ln x) = (\ln x + 1)\cos(x \ln x) \end{array}$$

(ii)

$$f(x) = e^{(e^x)} \Rightarrow (e^x)' e^{(e^x)} = e^x e^{(e^x)} = f'(x)$$

(iii)

$$\begin{array}{rclcrcl} f(x) & = & x^x & \Rightarrow & f'(x) & = & ? \quad \text{no rule applicable} \\ & = & e^{x \ln x} & \Rightarrow & f'(x) & = & (\ln x + 1)\, x^x \end{array}$$

$\Rightarrow$ Discussion of the functions we will have in the exercises!!