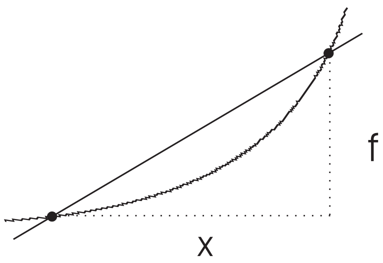
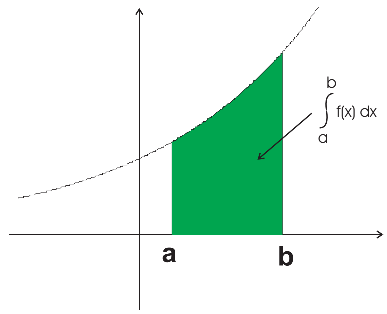
3.1 Recapitulation: Derivatives and Integrals

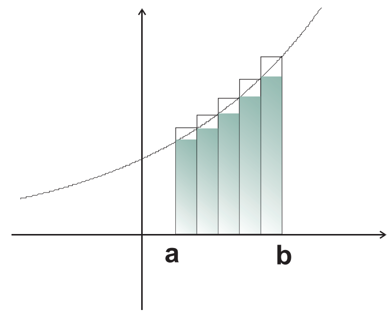
<u>Function:</u> $f(x), x \in \mathbb{R}, f(x) \in \mathbb{R}$; function f should behave "normal"



slope of secant →
slope of tangent



integral



approximation of integral

$$f'(x) \quad - \quad \text{slope of the tangent at the point } x$$
$$\int_a^b f(x)\mathrm{d}x \quad - \quad \text{area "under" } f(x) \text{ between } a \text{ and } b$$
$$\Rightarrow \quad \text{geometric definitions!}$$

Definition 30 *Derivative of f at the point x:*

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \overbrace{\frac{df}{dx}}^{\text{Leibniz notation}} \quad \hat{=} \frac{df}{dx}\text{-ratio of differentials}$$
$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{f(x)-f(x_0)}{x-x_0} \hat{=} \text{ratio of differences}$$

<u>Example</u>

$$f(x) = x^3; \qquad f(x+h) = (x+h)^3 = x^3 + 3hx^2 + 3h^2x + h^3$$
$$\frac{f(x+h)-f(x)}{h} = 3x^2 + 3hx + h^2 \rightarrow 3x^2 \text{ if } h \to 0$$
$$\Rightarrow f'(x) = 3x^2$$
$$\Rightarrow \quad \text{tedious work, rules will simplify all this considerably!!}$$

Definition 31 *Integral of f between the two points a and b, $a \times b$*

$$\int\limits_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta x\, f(a+k\Delta x) \qquad \text{with } \Delta x = \frac{b-a}{n}$$
$$= \left(\lim_{n\to\infty}\sum_{k=0}^{n-1}\Delta x\, f(a+k\Delta x)\right)$$
$$\sum^{n} > \int\limits_a^b > \sum^{n-1} \rightarrow \text{"Riemann-Integral"}$$

<u>In general:</u> Δx don't have to be the same for all k.

<u>Example:</u>

$$f(x) = x^3 \quad a=0,\ b=1 \rightarrow \Delta x = \frac{1}{n}$$
$$f(a+\frac{k}{n}) = (a+\frac{k}{n})^3 = a^3 + 3\frac{k}{n}a^2 + 3\frac{k^2}{n^2}a + \frac{k^3}{n^3}$$

$$\rightarrow \qquad \sum_{k=1}^{n} \Delta x\, f(x + k\Delta x) = \sum_{k=1}^{n} (\frac{1}{n}a^3 + 3\frac{k}{n^2}a^2 + 3\frac{k^2}{n^3}a + \frac{k^3}{n^4})$$

$\rightarrow$ difficult to perform further: we take: $a = 0$

$$\left(\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\right)$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)}{4}$$

$$\rightarrow \sum_{k=1}^{n} \Delta x\, f(a + k\Delta x) = \sum_{k=1}^{n} \frac{k^3}{n^4} = \frac{n^2(n+1)^2}{n^4 4} = \frac{(n+1)^2}{n^2}\frac{1}{4}$$

$$\stackrel{n\rightarrow\infty}{\rightarrow} \frac{1}{4} \Rightarrow \int_0^1 x^3 dx = \frac{1}{4}$$

$\Rightarrow$ tedious work, integration rules will help but it is still difficult