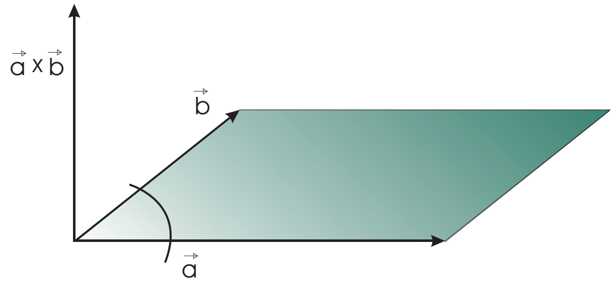
2.15 Vector product

The vector product as an operator on two vectors is only defined 3D:



$\vec{a},\vec{b} \in \mathbb{R}^3 \rightarrow \vec{a}\times\vec{b}$- vector which is

(i) perpendicular to $\vec{a}$ and $\vec{b}$

(ii) length is equal to the area of the parallelogram given by $\vec{a}$ and $\vec{b}$ $\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\,|\vec{b}|\sin\angle(\vec{a},\vec{b})$

in components this means:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \vec{a}\times\vec{b}$$

The definition of the vector product using the properties of the determinant will here be discussed only for 3D (however this approach allows to generalize the vector product to all dimensions larger 1D):

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \vec{a}\times\vec{b} &= \begin{vmatrix} \tilde{\mathbf{e}}_\mathbf{1} & a_1 & b_1 \\ \tilde{\mathbf{e}}_\mathbf{2} & a_2 & b_2 \\ \tilde{\mathbf{e}}_\mathbf{3} & a_3 & b_3 \end{vmatrix} \text{ the first "\textbf{bold}" column represents the formal determinants} \\ &= \vec{e}_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - \vec{e}_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + \vec{e}_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{aligned}$$

Note:

$$\begin{aligned} \vec{a}\times\vec{b} &= -\vec{b}\times\vec{a} \\ \vec{a}\times\vec{a} &= 0 \end{aligned}$$

$$\vec{a},\vec{b} \neq \vec{0} \quad \vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a},\vec{b} \text{ are linear dependent}$$

Example:

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \vec{a}\times\vec{b} = \begin{pmatrix} 2\cdot 6 - 3\cdot 5 \\ 3\cdot 4 - 1\cdot 6 \\ 1\cdot 5 - 2\cdot 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

Test:

$$\begin{aligned} \vec{a}\cdot\left(\vec{a}\times\vec{b}\right) &= -3+12-9 &= 0 \\ \vec{b}\cdot\left(\vec{a}\times\vec{b}\right) &= -12+30-18 &= 0 \end{aligned} \rightarrow \begin{matrix} \vec{a} & \perp & \vec{a}\times\vec{b} \\ \vec{b} & \perp & \vec{a}\times\vec{b} \end{matrix}$$

scalar+vector product:

$$\begin{aligned} \vec{a}\times\left(\vec{b}\times\vec{c}\right) &= \vec{b}\left(\vec{a}\cdot\vec{c}\right) - \vec{c}\left(\vec{a}\cdot\vec{b}\right) \\ \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \hat{=} \text{Volume of} \end{aligned}$$

So obviously the determinant of the 3 vectors (written as a 3×3 matrix) calculates the volume spanned by the 3 vectors.